NOTES ON DEAN'S PROBLEM

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Dick Dean asked if a free lattice could have an ascending chain of sublattices all isomorphic to $\mathbf{FL}(3)$. This is apparently an old problem, but I was not aware of it. Note that since $\mathbf{FL}(\omega)$ is a sublattice of $\mathbf{FL}(3)$, if one free lattice has such a chain then every $\mathbf{FL}(X)$ with $|X| \geq 3$ does.

Of course $\mathbf{FL}(3)$ has a proper sublattice isomorphic to itself and this gives a descending chain of sublattices isomorphic to $\mathbf{FL}(3)$. But note that the generators of the larger lattices are subelements of the generators of the smaller ones. The nonexistence of an ascending chain could be proved by showing that if both a, b, c and a', b', c' generate sublattices isomorphic to $\mathbf{FL}(3)$ and a, b, and c are in the sublattice generated by a', b', and c', then a', b', and c' occur as subterms of a, b, and c. We will give an example showing that this is not the case.

Given a, b, and c the next lemma, which is Lemma 9.13 of [1], limits where we might find the a', b', and c'.

Lemma 1. *Let σ be an isomorphism of $\mathbf{FL}(X)$ into $\mathbf{FL}(Y)$ and let $w \in \mathbf{FL}(X)$. If w_1 is a canonical joinand of w and $w_1 \notin X$, then $\sigma(w_1)$ is a canonical joinand of $\sigma(w)$.*

This lemma implies that if we look at the term tree of a each path from a to a leaf (i.e., a generator) there is a subterm u and an immediate subterm v of it such either a', b', or c' lies between u and v. Another way of looking at this is that the term in a', b', and c' giving a is just a "truncation" of the term in the free generators giving a.

The simpliest thing one might try is to write each of a, b and c as a join and try to place the a', b' and c' between these joinands and the element. But it is easy to argue that this is not possible. The next easiest possibility is illustrated in Figure 1. In this figure a_1 represents the join of some of the canonical joinands of a (those below a'). Similarly a_{21} is the meet of those canonial meetands of a_2 above c'. But, by Lemma 1, a_2 is a canonical joinand of a.

Of course this implies, for example, that $a_1 \vee b_2 \vee c_2 \leq a \wedge b_{21} \wedge c_{22}$. This suggests we try to find elements a_1, a_{21}, a_{22}, b_1, b_{21}, b_{22}, c_1, c_{21}, c_{22} of

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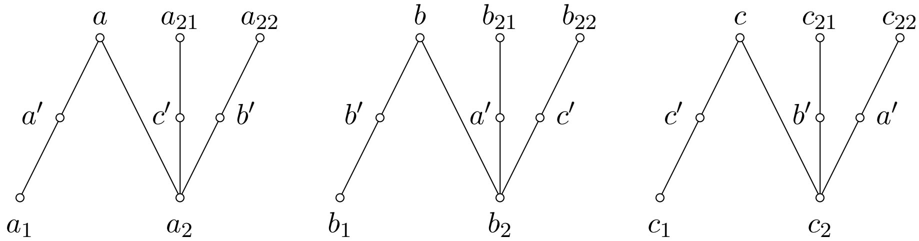

FIGURE 1.

$\mathbf{FL}(X)$ which satisfy the following relations:

(1)	$a_{21} \geq c_1$	$b_{21} \geq a_1$	$c_{21} \geq b_1$
(2)	$a_{22} \geq b_1$	$b_{22} \geq c_1$	$c_{22} \geq a_1$
(3)	$a_{21} \geq b_{21} \wedge b_{22}$	$b_{21} \geq c_{21} \wedge c_{22}$	$c_{21} \geq a_{21} \wedge a_{22}$
(4)	$a_{22} \geq c_{21} \wedge c_{22}$	$b_{22} \geq a_{21} \wedge a_{22}$	$c_{22} \geq b_{21} \wedge b_{22}$

These relations are illustrated in Figure 2.

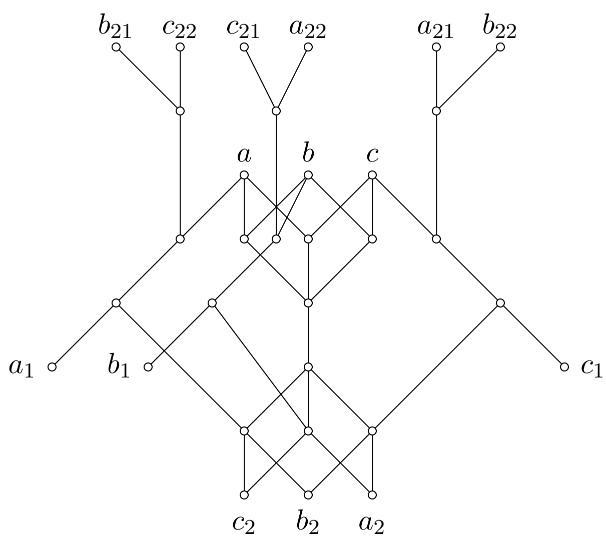

FIGURE 2.

My LISP file `solutions.lisp` finds more and more general solutions to equations such as (1)–(4). By a solution we mean a map from

$$S = \{a_1, a_{21}, a_{22}, b_1, b_{21}, b_{22}, c_1, c_{21}, c_{22}\}$$

into a free lattice such that (1)–(4) hold. We will take the free lattice to have generating set S so that the map is an endomorphism.

The file `dean.fp` has these relations. Applying `solutions.lisp` (twice) gives a solution $\overline{a}_1$, etc. Let

$$\begin{aligned}\overline{a}_1 = a_1 \vee [&(a_1 \vee b_{21} \vee c_{22}) \wedge (c_1 \vee a_{21} \vee b_{22})\\ &\wedge (a_1 \vee b_1 \vee a_{22} \vee b_{21} \vee c_{21}) \wedge (b_1 \vee c_1 \vee a_{22} \vee b_{22} \vee c_{21})\\ &\wedge (a_1 \vee b_1 \vee c_1 \vee a_{21} \vee b_{21} \vee c_{21}) \wedge (a_1 \vee b_1 \vee c_1 \vee a_{22} \vee b_{22} \vee c_{22})]\\ \vee [&(a_1 \vee b_{21} \vee c_{22}) \wedge (b_1 \vee a_{22} \vee c_{21})\\ &\wedge (a_1 \vee c_1 \vee a_{21} \vee b_{22} \vee c_{22}) \wedge (b_1 \vee c_1 \vee a_{21} \vee b_{22} \vee c_{21})\\ &\wedge (a_1 \vee b_1 \vee c_1 \vee a_{21} \vee b_{21} \vee c_{21}) \wedge (a_1 \vee b_1 \vee c_1 \vee a_{22} \vee b_{22} \vee c_{22})]\end{aligned}$$

$$\begin{aligned}\overline{a}_{21} = a_{21} \vee [&(a_{21} \vee b_{22} \vee c_1) \wedge (a_1 \vee a_{21} \vee b_{21} \vee c_1 \vee c_{22})\\ &\wedge (a_{22} \vee b_1 \vee b_{22} \vee c_1 \vee c_{21}) \wedge (a_1 \vee a_{21} \vee b_1 \vee b_{21} \vee c_1 \vee c_{21})\\ &\wedge (a_1 \vee a_{22} \vee b_1 \vee b_{21} \vee c_1 \vee c_{21}) \wedge (a_1 \vee a_{22} \vee b_1 \vee b_{21} \vee c_1 \vee c_{22})\\ &\wedge (a_1 \vee a_{22} \vee b_1 \vee b_{22} \vee c_1 \vee c_{22})\end{aligned}$$

$$\begin{aligned}\overline{a}_{22} = a_{22} \vee [&(a_{22} \vee b_1 \vee c_{21}) \wedge (a_1 \vee a_{22} \vee b_1 \vee b_{21} \vee c_{22})\\ &\wedge (a_{21} \vee b_1 \vee b_{22} \vee c_1 \vee c_{21}) \wedge (a_1 \vee a_{21} \vee b_1 \vee b_{21} \vee c_1 \vee c_{21})\\ &\wedge (a_1 \vee a_{21} \vee b_1 \vee b_{21} \vee c_1 \vee c_{22}) \wedge (a_1 \vee a_{21} \vee b_1 \vee b_{22} \vee c_1 \vee c_{22})\\ &\wedge (a_1 \vee a_{22} \vee b_1 \vee b_{22} \vee c_1 \vee c_{22})\end{aligned}$$

The elements $\overline{b}_1$, etc., are given by applying the permutation $a \mapsto b \mapsto c \mapsto a$ to the above.

Define $a = \overline{a}_1 \vee (\overline{a}_{21} \wedge \overline{a}_{22})$. Simplifying into canonical form:

$$\begin{aligned}a = a_1 \vee [&(a_1 \vee b_{21} \vee c_{22}) \wedge (c_1 \vee a_{21} \vee b_{22})\\ &\wedge (a_1 \vee b_1 \vee a_{22} \vee b_{21} \vee c_{21}) \wedge (b_1 \vee c_1 \vee a_{22} \vee b_{22} \vee c_{21})\\ &\wedge (a_1 \vee b_1 \vee c_1 \vee a_{21} \vee b_{21} \vee c_{21}) \wedge (a_1 \vee b_1 \vee c_1 \vee a_{22} \vee b_{22} \vee c_{22})]\\ \vee [&(a_1 \vee b_{21} \vee c_{22}) \wedge (b_1 \vee a_{22} \vee c_{21})\\ &\wedge (a_1 \vee c_1 \vee a_{21} \vee b_{22} \vee c_{22}) \wedge (b_1 \vee c_1 \vee a_{21} \vee b_{22} \vee c_{21})\\ &\wedge (a_1 \vee b_1 \vee c_1 \vee a_{21} \vee b_{21} \vee c_{21}) \wedge (a_1 \vee b_1 \vee c_1 \vee a_{22} \vee b_{22} \vee c_{22})]\\ \vee [&(c_1 \vee a_{21} \vee b_{22}) \wedge (b_1 \vee a_{22} \vee b_{21})\\ &\wedge (a_1 \vee c_1 \vee a_{21} \vee b_{22} \vee c_{22}) \wedge (a_1 \vee b_1 \vee a_{22} \vee b_{21} \vee c_{22})\\ &\wedge (a_1 \vee b_1 \vee c_1 \vee a_{21} \vee b_{21} \vee c_{21}) \wedge (a_1 \vee b_1 \vee c_1 \vee a_{22} \vee b_{22} \vee c_{22})]\end{aligned}$$

The formulas for b and c are obtained by applying the above permutation.[1]

[1]Looking at Figure 1 we see that $a \geq a' \geq b_2$. Defining $a = \overline{a}_1 \vee (\overline{a}_{21} \wedge \overline{a}_{22})$, as we did above, does not guarentee this will hold. In fact the first solution to (1)–(4) does not satisfy this but the second one does. This was lucky.

If we choose

$$a' = a \wedge \overline{b}_{21} \wedge \overline{c}_{22} = a \wedge (a_1 \vee b_{21} \vee c_{22}),$$

etc., then both $\{a, b, c\}$ and $\{a', b', c'\}$ generate a sublattice isomorphic to $\mathbf{FL}(3)$ with the later sublattice properly containing the former. Moreover, a' is more complex that a, etc. In fact we can choose any $a' \in a \wedge \overline{b}_{21} \wedge \overline{c}_{22} / \overline{a}_1 \vee \overline{b}_2 \vee \overline{c}_2$ and independently choose similar b' and c'. Thus a', b', and c' can be choosen to be arbitrarily complex.

1. An almost proof that an ascending chain cannot exist

As noted above, whenever the sublattice generated by a, b, and c is contained that generated by a', b', and c' and both are free lattices, then the primed generators determine a truncation of the term tree of a, b, and c. This means that for each path of the tree there is an element and an immediate subelement such that one of a', b' or c' lies in the interval between the element and the subelement. There are only finitely many possible truncations even counting the labelling. Thus we we had a ascending chain $\mathbf{Sb}(a^k, b^k, c^k)$ of free sublattices we could assume that the trucation pattern of $\{a^0, b^0, c^0\}$ determined by $\{a^k, b^k, c^k\}$ is the same for all k.

The next lemma almost shows this is impossible.

Lemma 2. *Suppose $a^- \leq a^+$, $b^- \leq b^+$, and $c^- \leq c^+$ are strongly join and meet irredundant, i.e., $a^- \not\leq b^+ \vee c^+$ and symmetrically and dually. If $a^- \leq a \leq a^+$ and similarly for b and c, then a is the only element of the sublattice generated by a, b, and c in a^+/a^-.*

Proof. Suppose $a^- \leq w$ and w is in the sublattice generated by a, b, and c. Then either $w \geq a$ or $w \leq b \vee c$. The latter gives

$$a^- \leq w \leq b \vee c \leq b^+ \vee c^+,$$

a contradiction. So $w \geq a$ and similarly $w \leq a$. □

References

[1] R. Freese, J. Ježek, and J. B. Nation, *Free Lattices*, Amer. Math. Soc., Providence, 1995, Mathematical Surveys and Monographs, vol. 42.

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