NOTES ON DIRECT DECOMPOSITIONS

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Lemma 1. *Let* $\mathbf{L}$ *be a modular lattice of equivalence relations, and let* α, α', *and* $\beta \in L$. *If* α *and* α' *permute and* $\alpha \wedge \alpha' \leq \beta \leq \alpha'$ *then* α *and* β *permute.*

Proof. Let $a \; \alpha \; b \; \beta \; c$. Then $\langle a, c\rangle \in \alpha \vee \beta \leq \alpha \vee \alpha'$. Hence there is a b' such that the relations of Figure 1 hold.

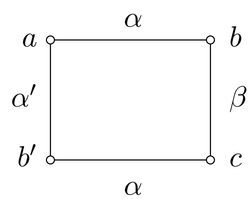

Figure 1.

Hence $\langle a, b'\rangle \in \alpha' \wedge (\alpha \vee \beta) = (\alpha \wedge \alpha') \vee \beta = \beta$, showing that $\alpha \circ \beta = \beta \circ \alpha$. □ □

Lemma 2. *Let* $\mathbf{L}$ *be a lattice of equivalence relations, and let* α, α', *and* $\beta \in L$. *If* α *and* α' *permute and* $\alpha \leq \beta \leq \alpha \vee \alpha'$ *then* α' *and* β *permute.*

Proof. Easy. □ □

Theorem 3. *Let* $\mathbf{L}$ *be a finite dimensional modular lattice of equivalence relations with elements* α, α', β, *and* β' *satisfying*

$$\begin{aligned} \alpha \vee \alpha' = \beta \vee \beta' = \alpha \vee \beta' = \alpha' \vee \beta = 1 \\ \alpha \wedge \alpha' = \beta \wedge \beta' = \alpha \wedge \beta' = \alpha' \wedge \beta = 0. \end{aligned} \tag{1}$$

Moreover, assume that

$$\alpha \circ \alpha' = \alpha' \circ \alpha \qquad \beta \circ \beta' = \beta' \circ \beta. \tag{2}$$

If there is no homomorphism of the sublattice of $\mathbf{L}$ *generated by* $\{\alpha, \beta, \alpha', \beta'\}$ *onto* $\mathbf{M}_4$, *then* α *and* β' *permute.*

Note by this comment, it is clear that a somewhat stronger theorem for algebras is true. Namely, we need only assume that the sublattice of **Con L** generated by α, α', β, and β' is finite

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Proof. Clearly, in order to prove this theorem, we may assume that $\mathbf{L}$ is generated by $\{\alpha, \beta, \alpha', \beta'\}$. By a dimension argument, if every pair of generators met to 0, then every pair would join to 1. This would make $\mathbf{L} \cong \mathbf{M}_4$, contrary to the hypothesis. Thus, without loss of generality, $\alpha' \wedge \beta' > 0$. Let

$$\alpha_1 = \alpha \vee (\alpha' \wedge \beta') \qquad \beta_1 = \beta \vee (\alpha' \wedge \beta').$$

It is easy to check that α_1, β_1, α', and β' still satisfy (1) with $\alpha' \wedge \beta'$ in place of 0. By Lemma 2, α_1 and α' permute as do β_1 and β'. Let $\mathbf{L}_1$ be the sublattice generated by α_1, β_1, α', and β'. We claim that $\mathbf{M}_4$ is not a homomorphic image of $\mathbf{L}_1$.

To see this consider the homomorphism $f : \mathbf{FL}(x, y, x', y') \twoheadrightarrow \mathbf{M}_4$. Let $f(x) = a$, $f(y) = b$, $f(x') = a'$, and $f(y') = b'$, where a, b, a', and b' are the atoms of $\mathbf{M}_4$. Define maps α_n and $\beta_n : \{a, b, a', b'\} \rightarrowtail \mathbf{FL}(x, y, x', y')$ for $n \geq 0$ by $\beta_0(a) = x$ and

$$\beta_{n+1}(a) = x \wedge [\beta_n(b) \vee \beta_n(a')] \wedge [\beta_n(b) \vee \beta_n(b')] \wedge [\beta_n(a') \vee \beta_n(b')]. \tag{3}$$

(Do not confuse the lower maps, denoted β_n with our specific elements β and β_1.) The definition of β_n on b, a', and b' is symmetric. α_n is defined dually. By McKenzie [1], $f(z) \geq a$ if and only if $z \geq \beta_n(a)$ for some n. From this we get the following lemma.

Lemma 4. *If $\mathbf{M}$ is a lattice generated by x, y, x', and y' then the map $g(x) = a$, $g(y) = b$, $g(x') = a'$, and $g(y') = b'$ can be extended to a homomorphism of $\mathbf{M}$ onto $\mathbf{M}_4$ if and only if the following hold in $\mathbf{M}$.*

$$\beta_n(a) \not\leq \alpha_n(b) \qquad \textit{for all } n \geq 0 \tag{4}$$

We wish to apply this lemma to $\mathbf{L}$. For each n, $\beta_n(a)$ is a term in the language of lattices. We let $\beta_n^{\mathbf{L}}(a)$ be the interpretation of this term in $\mathbf{L}$ under the subsitiution $x = \alpha$, $y = \beta$, $x' = \alpha'$, and $y' = \beta'$. Moreover, we let $\beta_n^{\mathbf{L}_1}(a)$ be the interpretation of this term in $\mathbf{L}_1$ under the subsitiution $x = \alpha_1$, $y = \beta_1$, $x' = \alpha'$, and $y' = \beta'$.

Since our $\mathbf{L}$ does not have $\mathbf{M}_4$ as a homomorphic image, there is an n such that the relation

$$\beta_n^{\mathbf{L}}(a) \leq \alpha_n^{\mathbf{L}}(b)$$

holds in $\mathbf{L}$. Using (1) and some modular calculations, one can show that

$$\beta_n^{\mathbf{L}_1}(a) = \beta_n^{\mathbf{L}}(a) \vee (\alpha' \wedge \beta'). \tag{5}$$

(Sketch: First show that if $\beta_{n+1}(a)$ is defined inductively as

$$\beta_{n+1}(a) = x \wedge [\beta_n(a') \vee \beta_n(b')]$$

instead of as it was defined (3) (and similarly for $\beta_n(a')$, etc.), the interpretation in any modular lattice satisfying (1) is the same. Then note the following which holds in all lattices,

$$\beta_{n+1}(a) = \beta_n(a) \wedge [\beta_n(a') \vee \beta_n(b')]$$

Now use the modular law to specifically get a "one sided" form and use this to derive (5).) Hence

$$\begin{aligned}\beta_n^{\mathbf{L}_1}(a) &= \beta_n^{\mathbf{L}}(a) \vee (\alpha' \wedge \beta') \\ &\leq \alpha_n^{\mathbf{L}}(b) \vee (\alpha' \wedge \beta') \\ &\leq \alpha_n^{\mathbf{L}_1}(b).\end{aligned}$$

Thus by Lemma 4, $\mathbf{L}_1$ does not have $\mathbf{M}_4$ as a homomorphic image. Thus, by induction on the length of the lattice, we conclude that α_1 and β' permute. Now using this and an application of Lemma 1 we calculate

$$\begin{aligned}\alpha \vee \beta' &= \alpha_1 \vee \beta' \\ &= \alpha_1 \circ \beta \\ &= [\alpha \vee (\alpha' \wedge \beta')] \circ \beta' \\ &= [\alpha \circ (\alpha' \wedge \beta')] \circ \beta' \\ &= \alpha \circ \beta'\end{aligned}$$

completing the proof. □ □

The theorem does have content. There are infinitely many 4-generated, finite dimensional, subdirectly irreducible modular lattices which satisfy (1). In fact there is one for each possible length. The one of length 4 is diagrammed in Figure 2.

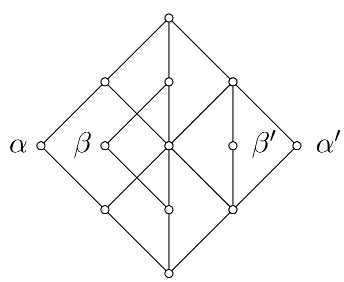

Figure 2.

References

[1] R. McKenzie, *Equational bases and non-modular lattice varieties*, Trans. Amer. Math. Soc. **174** (1972), 1–43.

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