ON EXTENDING LEMMA 4

Ralph Freese

In §5.2 and §5.3 of [4] the authors present two famous direct decomposition theorems. The Birkhoff-Ore Theorem states that an algebra **A** with permuting congruences and a one-element subalgebra has unique factorization as a direct product of directly indecomposable algebras provided that **Con A** is finite dimensional. Jónsson was able to prove a similar theorem, only assuming that **Con A** is modular, provided **A** is finite. (Without the assumption of a one-element subalgebra, ones gets isotopic versions of both of these theorems, see [4].) A very clear proof, emphasizing the common aspects of these theorems, is given in [4]. The authors ask if these theorems have a common generalization: *does an algebra with a finite dimensional modular congruence lattice have unique factorization?* Lemma 4 of §5.2 of [4] shows exactly where the finiteness of **A** is used in Jónsson's result to prove certain congruences permute. The authors point out that if this lemma could be proved under the weaker assumption that **Con A** is a finite dimensional modular lattice then the desired generalization would be valid.

Lemma 4 concerns four congruences, α, α', β, and β', such that α and α' form a complementary permuting pair as do β and β'. Moreover,

$$\alpha \wedge \beta' = \alpha' \wedge \beta = 0.$$

The desired conclusion is that α and β' also permute. In [1] we have been able to prove this under the assumption that the sublattice of **Con A** generated by these four elements does not have $\mathbf{M}_4$ as a homomorphic image.

However in this note we construct an algebra which shows that the full generalization of Lemma 4 is not valid.

Preliminaries. Let **G** be a group and let S be a set of endomorphism of **G**. Assume that **G** has no S-invariant subgroups. We define a unary algebra

$$\mathbf{A} = \langle G \times G, f_{a,b,s} \rangle_{a,b \in G,\, s \in S}$$

where

$$f_{a,b,s}(x, y) = \langle as(x), s(y)b \rangle.$$

This research was partially supported by NSF grant no. DMS89–01756

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Theorem 1 (Gumm [2], [3]). *For the algebra* $\mathbf{A}$ *defined above, we have the following.* Theorem 'gumm'

(1) *Both the projections kernels,* η_0 *and* η_1*, are congruences on* $\mathbf{A}$*.*
(2) *If* σ *is an automorphism of* $\mathbf{G}$ *which respects, i.e. commutes with, every element of* S*, then*

$$\overline{\sigma} = \{\langle\langle a, b\rangle, \langle c, d\rangle\rangle : a\sigma(b) = c\sigma(d)\}$$

is a congruence on $\mathbf{A}$*. Moreover, every element of* $\mathbf{Con\ A}$*, except* $0_{\mathbf{A}}$*,* $1_{\mathbf{A}}$*,* η_0*, and* η_1*, has this form.*
(3) $\mathbf{Con\ A} \cong \mathbf{M}_n$*, where* n *is two more than the cardinality of the set of automorphisms of* $\mathbf{G}$ *which respect* S*.*
(4) η_i *permutes with every element of* $\mathbf{Con\ A}$*, for* $i = 0, 1$*.*

Proof. It is straightforward to verify that $\overline{\sigma}$ is a congruence and that if σ and τ are distinct, then $\overline{\sigma} \neq \overline{\tau}$. The proof of the second statement of (2) is given in Lemma 3.4 of [2], which Gumm credits to Wolk. The other parts of the theorem are either elementary or follow from (2). □

The next lemma tells when $\overline{\sigma}$ and $\overline{\tau}$ permute.

Lemma 2. *Let* σ *and* τ *be distinct automorphisms of* $\mathbf{G}$ *which respect* S*.* Theorem 'perm'

(1) *The congruences* $\overline{\sigma}$ *and* $\overline{\tau}$ *permute if and only if the map* $x \mapsto \sigma(x)^{-1}\tau(x)$*, from* G *to* G*, is onto.*
(2) *The map* $x \mapsto \sigma(x)^{-1}\tau(x)$ *is always one-one.*
(3) *If* $\mathbf{G}$ *is finite, then* $\overline{\sigma}$ *and* $\overline{\tau}$ *permute.*
(4) *If* $\mathbf{G}$ *is abelian, then* $\overline{\sigma}$ *and* $\overline{\tau}$ *permute.*

Proof. Since σ and τ are distinct, so are $\overline{\sigma}$ and $\overline{\tau}$. Thus $\overline{\sigma} \vee \overline{\tau} = 1$. Hence $\overline{\sigma}$ and $\overline{\tau}$ permute if and only if for all a, b, c, $d \in G$ there are x, $y \in G$ such that $\langle a, b\rangle\ \overline{\sigma}\ \langle x, y\rangle\ \overline{\tau}\ \langle c, d\rangle$. This holds if and only if

$$a\sigma(b) = x\sigma(y) \qquad x\tau(y) = c\tau(d).$$

Eliminating x from these equations, we see that $\overline{\sigma}$ and $\overline{\tau}$ permute if and only if there is a y such that

$$\sigma(y)^{-1}\tau(y) = \sigma(b)^{-1}a^{-1}c\tau(d).$$

since the right side of the above equation can represent any element of G, (1) follows.

The second part follows from the fact that $\{x \in G : \sigma(x) = \tau(x)\}$ is an S-invariant subgroup of $\mathbf{G}$. (3) follows from (2). When $\mathbf{G}$ is abelian, $\{\sigma(x)^{-1}\tau(x) : x \in G\}$ is also an S-invariant subgroup of $\mathbf{G}$, and hence must be all of $\mathbf{G}$. □

The example. Let $\mathbf{H}$ be a simple, nonabelian group. Let $\mathbf{G}$ be the direct sum of infinitely many copies of $\mathbf{H}$ indexed by $\mathbb{Z}$, *i.e.*,

$$G = \{f \in H^{\mathbb{Z}} : f(i) = 1 \text{ for all but finitely many } i\}.$$

Let σ be the shift automorphism of $\mathbf{G}$, *i.e.*, $(\sigma f)(i) = f(i-1)$. Let T be the set of automorphisms on $\mathbf{G}$ arising from inner automorphisms of $\mathbf{H}$, *i.e.*, each $s \in T$ has the form $(sf)(i) = x^{-1}f(i)x$ for some $x \in H$ independent of i. Let $S = T \cup \sigma$.

Lemma 3. $\mathbf{G}$ *has no nontrivial subgroups invariant under* S. Theorem 'lemma1'

Proof. We need to show that if $f \in G$ is not 1, then the S-subgroup of $\mathbf{G}$ generated by f is $\mathbf{G}$. We prove this by induction on the size of the support of f. If the support of f is 1, say $f(i) = 1$ for all $i \neq 0$ but $f(0) \neq 1$, then using the inner automorphisms and the fact that $\mathbf{H}$ is simple, we obtain elements whose 0^{th} coordinate is arbitrary and whose other coordinates are all 1. Using σ we can move this to any coordinate and by using multiplication we can generate an arbitrary element in G.

Now suppose that the support of f is at least 2. We may assume that the support of f lies in $\{0, 1, \ldots, n-1\}$ and that $f(0) \neq 1$. Let k be the next nonidentity coordinate. Again since $\mathbf{H}$ is simple, we can generate g from f such that $g(0)$ is arbitrary. In particular, we may assume that $g(0)$ does not commute with $f(k)$. Let $h = [f, \sigma^k(g)]$. Then $h(k) = f(k)^{-1}g(0)^{-1}f(k)g(0) \neq 1$ but $h(i) = 1$ for $i < k$ and for $i \geq n$. Thus h has strictly smaller support and so we are done by induction. □

Two more observations. First σ is a group automorphism of $\mathbf{G}$ which also respects the unary operations S, *i.e.*, σ commutes with every $s \in S$. Second the map $f \mapsto f^{-1}\sigma(f)$ is not onto. Indeed, if the coordinates of an element of the form $f^{-1}\sigma(f)$ are multiplied together backwards, the answer is 1.

Thus combining these facts we see that Lemma 4 of §5.2 of [4] does not hold under the weaker hypothesis that **Con A** is a finite dimensional modular lattice. Indeed, let $\mathbf{A}$ is the algebra constructed above and let τ is the identity automorphism on $\mathbf{G}$. Then, if we let $\alpha = \overline{\sigma}$, $\beta = \eta_1$, $\alpha' = \eta_0$, and $\beta' = \overline{\tau}$, every pair of these elements join to $1_{\mathbf{A}}$ and meet to $0_{\mathbf{A}}$ and α and α' permute as do β and β'. However, α and β' do not permute.

On the other hand it follows easily from Lemma 7 of §5.2 that the four algebras, $\mathbf{A}/\alpha$, $\mathbf{A}/\alpha'$, $\mathbf{A}/\beta$, and $\mathbf{A}/\beta'$, are pairwise modular isotopic. Thus this example does not answer the problem of §5.3, even for the isotopic case.

References

1. R. Freese, *Notes on direct decompositions*, preprint.
2. H. P. Gumm, *Is there a Mal'cev theory for single algebras?*, Algebra Universalis **8** (1978), 320–329.
3. H. P. Gumm, *Algebras in congruence permutable varieties: Geometrical properties of affine algebras*, Algebra Universalis **9** (1979), 8–34.
4. Ralph McKenzie, George McNulty and Walter Taylor, *Algebras, Lattices, Varieties, Volume I*, Wadsworth and Brooks/Cole, Monterey, California, 1987.

University of Hawaii, Honolulu, HI 96822
E-mail address: ralph@math.hawaii.edu