Math 241 Final, Spring 2023

Name: ______________________________

Draw a circle around your section number below

	Instructor	TA	Recitation
1	Hugh Chou	Christin Sum	W 10:30-11:30
2	Hugh Chou	Christin Sum	W 11:30-12:20
3	Lyon Lanerolle	Katie Menssen	T 9:30-10:20
4	Lyon Lanerolle	Katie Menssen	T 12-12:50
5	Pui Hang Lee	Pui Hang Lee	F 12:30-1:20
6	Pierre-Olivier Parisé	Rukiyah Walker	F 9:30-10:20
7	Pierre-Olivier Parisé	Rukiyah Walker	F 10:30-11:20
8	Monir Zaman	Sydney Fields	Th 9:30-10:20
9	Monir Zaman	Sydney Fields	Th 12:30-1:20

Question	Points	Score
1	12	
2	5	
3	5	
4	8	
5	8	
6	8	
7	10	
8	6	
9	10	
10	5	
11	8	
12	5	
13	5	
14	5	
Total:	100	

- This exam has 14 printed pages.
- You may not use notes or calculators on the test.
- You may not use electronic devices or access the internet.
- Please ask if anything seems confusing or ambiguous.
- You must show all your work and make clear what your final solution is (e.g. by drawing a box around it).
- You have 2 hours to complete this exam.
- Good luck!

1. Calculate the following limits. Do not use l'Hôpital's rule (don't worry if you don't know what this is). If the limit is infinite, specify whether it is $+\infty$ or $-\infty$.

   (a) (4 points)

   $$\lim_{x\to 1} \frac{1-x^2}{x^2-3x+2}$$

   (b) (4 points)

   $$\lim_{x\to -3^+} \frac{|x-3|-5}{2x+6}$$

   (c) (4 points)

   $$\lim_{x\to 0} \frac{x-\tan 3x}{\sin 4x}$$

2. (5 points) Find the derivative of $y = \sqrt{3x+4}$ using the definition of the derivative as a limit. (Warning: You will not get credit if you use the rules of differentiation.)

3. Circle **True** or **False** for each question. No justification is required. No credit will be given for ambiguous responses.

   (a) (1 point) TRUE or FALSE Assume that functions f and g are positive, differentiable and strictly increasing on the interval $(0,1)$, then their product fg must be strictly increasing as well.

   (b) (1 point) TRUE or FALSE Suppose that c is a critical point for f and $f'(c)$ exists. Then f must have either a local maximum or local minimum at c.

   (c) (1 point) TRUE or FALSE The function defined by

$$f(x) = \begin{cases} 3 - x, & \text{if } x \le 2 \\ (x-2)^2 + 1, & \text{if } x > 2 \end{cases}$$

   is continuous at $x = 2$.

   (d) (1 point) TRUE or FALSE If $f'(x) = g'(x)$ for all values of x between 0 and 1 then $f(x) = g(x)$ for all values of x between 0 and 1.

   (e) (1 point) TRUE or FALSE If $\lim_{x\to 2} \frac{f(x)-5}{x-2} = 3$, then $f(2) = 5$.

4. (8 points) Farmer Brown needs to make three enclosures to separate her three types of livestock (she raises sheep, goats and pigs). Since she only has 1000m of fencing, she decides to make three adjacent pens, by fencing off a rectangular region and dividing it (using the same fencing) into three equal parts. Find the dimensions that maximize the combined area of the three enclosures.

   - *Your response should include a properly labeled, simple illustration of the enclosure.*
   - *You need to show your answer gives the maximum.*
   - *Your answer should involve calculus.*

5. Consider the function $f(x) = x^{5/3} - 5x^{2/3}$.

   (a) (4 points) Find the open intervals on which the graph of f is increasing and decreasing.

   (b) (4 points) Find the open intervals on which the graph of f is concave up and concave down.

6. The function $f(x) = \frac{3x^2}{x^2+27}$ has the following properties. (**You do not need to check these properties. In particular, you do not need to differentiate this function!**)

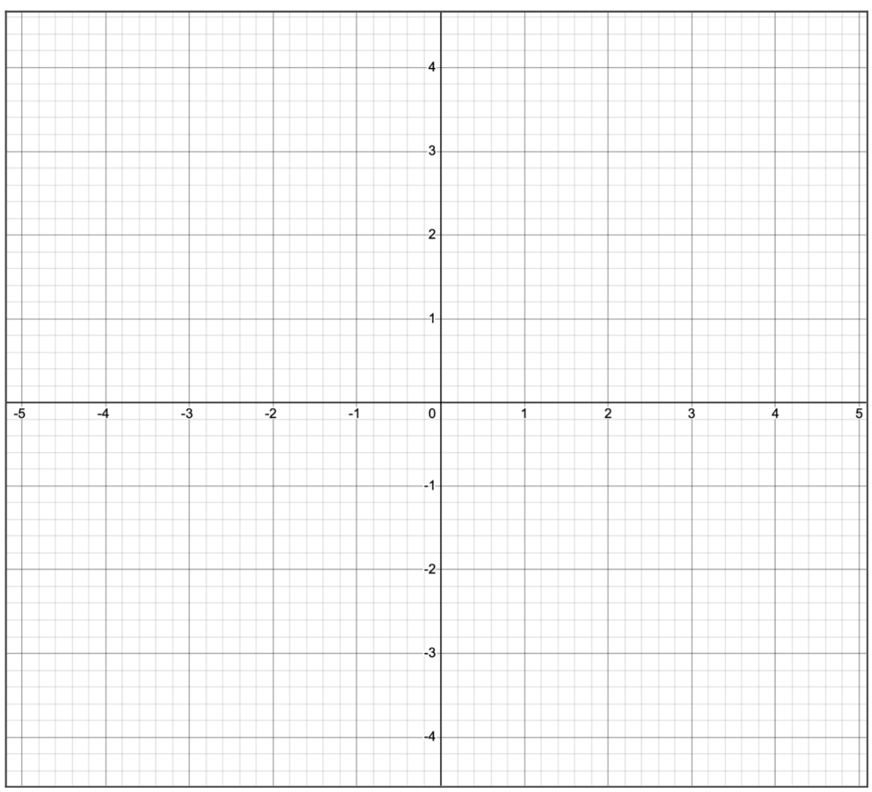
- f is non-negative on $(-\infty, \infty)$ (i.e. $f(x) \geq 0$ for all x)
- f is increasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$
- f is concave up on $(-3, 3)$ and concave down on $(-\infty, -3)$ as well as $(3, \infty)$.

(a) (3 points) Does the function have any asymptotes? If so, what are they?

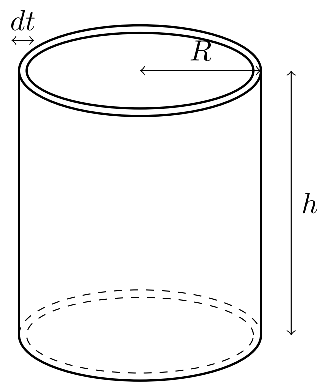
(b) (1 point) Does the graph have any local maxima or minima? If so, what are they?

(c) (1 point) Does the graph have any inflection points? If so, what are they?

(d) (3 points) On the axes below, sketch the function, incorporating the above information.



7. (a) (3 points) For the thin cylindrical shell pictured below, find the volume dV in terms of its radius R, height h and thickness dt.



$dV =$ ______________________________

(b) (7 points) Write an integral for the volume of the solid obtained by rotating the region bounded by $y = 0$ and $y = 4x - x^2$ about the line $x = -1$ by using the method of cylindrical shells.

- Your response should include a graph of the planar region begin rotated.
- Carefully decide whether the integral should involve dx or dy.
- **Simply set up the correct integral – do not evaluate it!**

8. (6 points) Compute the integral $\int_{-\frac{2\pi}{3}}^{8\pi} \sin(x/2)\mathrm{d}x$.

9. Compute each of the following integrals.

(a) (5 points) $\displaystyle\int_0^{1/2} \left|2x - \frac{1}{2}\right| \, \mathrm{d}x$

(b) (5 points) $\displaystyle\int \frac{x^5}{(x^3+1)^3} \, \mathrm{d}x$ (You do not need to simplify your answer.)

10. (5 points) Given that $f''(x) = 12(x-1)^2 - 1$, $f(1) = 1$ and $f(0) = -1$, find a formula for $f(x)$.

11. (8 points) Find the area of the region bounded by the graphs $y = (x-1)^3$ and $y = 2(x-1)^2$.

- Your response should include a graph of the planar region.
- Carefully set up an integral describing the area before evaluating it.

12. (5 points) A function $f(x)$ is defined by

$$f(x) = \int_{x^2}^{x} \sqrt{1+t}\,\mathrm{d}t.$$

Find $f'(x)$ by using the Fundamental Theorem of Calculus.

13. (5 points) Find the tangent line to the graph $x^2 + xy - y^2 = 1$ at the point $(2, 3)$.

14. (5 points) Let

$$f(x) = \sqrt{x - \frac{7}{x}}.$$

Use a linearization/linear approximation or a differential together with the fact that $f(4) = \frac{3}{2}$ to find an approximation to $f(4.1)$.