Calculus I (Math 241)
(In Progress)

The following is a collection of Calculus I (Math 241) problems. Students may expect that their final exam is comprised, more or less, of one problem from each section, or one that is similar. To the instructors the collection indicates what will be expected of the students in the common final.

The course will cover more material than what is reflected in the problems. In particular students shall develop a conceptual understanding for calculus in addition to learning how to solve problems as below. Still, being able to solve the problems (closed book and in a clear and transparent manner) will be an indication that you have a firm grasp of the course material.

On the exam no calculators will be allowed. If you insist on a decimal-numerique answer, then you may use a simple scientific calculator in your practice (not graphing and not symbolic). The solution for each problem has to include enough work to show how it was derived. If there are errors in the work, you may not receive credit, even if the final answer is correct.

1 Limits

Problem 1. Determine the following limits, or explain why they do not exist:

(a) $\displaystyle\lim_{x\to 2} \frac{x^2 - 8}{x - 2}$

(b) $\displaystyle\lim_{x\to \frac{\pi}{2}^+} \frac{\cot x}{x - \frac{\pi}{2}}$

(c) $\displaystyle\lim_{x\to 1^+} \frac{1}{\sqrt{x-1}} - \frac{1}{\sqrt{x^2-1}}$

(d) $\displaystyle\lim_{x\to 0} \frac{4x}{\tan 2x}$

(e) $\displaystyle\lim_{x\to 1^-} \frac{1 - \sqrt{2x-1}}{x^2 - 1}$

(f) $\displaystyle\lim_{x\to \infty} \frac{2\sqrt{x} + x^{-1}}{3x - 7}$

(g) $\displaystyle\lim_{x\to \infty} \frac{3}{2}\left(\frac{x}{x-1}\right)^{2/3}$

(h) $\displaystyle\lim_{x\to 0} \frac{\sin 3x}{5x}$

(i) $\displaystyle\lim_{x\to -1} \frac{x^2 - x - 2}{x + 1}$

(j) $\displaystyle\lim_{x\to 4} \frac{\sqrt{x} - 2}{x - 4}$

(k) $\displaystyle\lim_{x\to 0^-} x \csc 2x$

(l) $\displaystyle\lim_{x\to 4} \frac{4 - x}{2 - \sqrt{x}}$

(m) $\displaystyle\lim_{x\to \infty} \left[\frac{\sqrt{x^2 - 4} - \sqrt{x^2 + 27}}{x}\right]$

(n) $\displaystyle\lim_{x\to 1} \frac{\sin(1 - \sqrt{x})}{x - 1}$

2   Tangent Lines

Problem 2. Find the equation of the tangent line of the function $f(x)$ at the point $x = a$,

1. $f(x) = x^3 + 1$ at $a = 3/2$

2. $f(x) = \sin x$ at $a = \pi/6$

3. $f(x) = \cos^2 x$ at $a = \pi/3$

4. $f(x) = \tan x$ at $a = \pi/6$

5. $f(x) = \sqrt{x + 1}$ at $a = 3$

6. $f(x) = \dfrac{1}{x^2 + 1}$ at $a = 1$

7. $f(x) = x \sin x$; at $a = \pi/2$

8. $f(x) = x^4(x - 3)^2$; at $a = 1$

9. $f(x) = \sqrt{x} \tan x$; at $a = \pi/4$

10. $f(x) = \dfrac{x}{\sin x}$; at $a = \pi/2$

11. $f(x) = x^2 \cos x$; at $a = \pi/2$

3   Derivatives–First Principle

Problem 3. Differentiate the following functions at the given point using first principles (find the limit of the appropriate difference quotient):

1. $f(x) = x^3$ at $x = 2$

2. $f(x) = x^2 + x$ at $x = 3$

3. $f(x) = \sqrt{x}$ at $x = 4$

4. $f(x) = 1/x$ at $x = 2$

5. $f(x) = x^{\frac{3}{2}}$ at $x = 1$

6. $f(x) = \dfrac{1}{\sqrt{x}}$ at $x = 4$

7. $f(x) = x^{1.5}$ at $x = a$

8. $f(x) = x^{-2}$ at $x = a$

9. $f(x) = |x|$ at $x = -2$

10. $f(x) = x \cdot |x|$ at $x = 0$

4   Rules of Differentiation

Problem 4. Calculate the derivatives of five (5) of the following functions.

$$a(x) = (x^2 - 1)^7 \qquad\qquad\qquad b(x) = (x^3 - x)^2$$

$$c(x) = \frac{x^2 - 1}{x^3 + x} \qquad\qquad\qquad d(x) = \frac{\cos x}{x^2 + 1}$$

$$e(x) = x(x^2 + 1)^5 \qquad\qquad\qquad f(x) = \cos x(x^2 + 7)$$

$$g(x) = (x^2 + 1)\cos x \qquad\qquad\qquad h(x) = x\sec(2x)$$

$$i(x) = \frac{x\cos x}{x^2 + \tan x} \qquad\qquad\qquad j(x) = \frac{\sin x}{x^2 + 3}$$

$$k(x) = (x + 4)\cot x \qquad\qquad\qquad l(x) = \frac{x + 4}{2 + \cos x}$$

$$m(x) = \sqrt{2 + \cos x} \qquad\qquad\qquad r(x) = \sqrt{\tan^2(x - 1) + \cos x}$$

$$o(x) = x\sin(x^2 + 2) \qquad\qquad\qquad p(x) = (x^2 - 1)\tan(x - 4)$$

$$q(x) = \frac{\sin(x + 2)}{\cos(x - 3)} \qquad\qquad\qquad n(x) = \frac{1}{\sqrt{1 + \tan x}}$$

$$s(x) = \cot(\sin^2 x + x^3) \qquad\qquad\qquad t(x) = \sec(\sqrt{\cos(3x) + x})$$

$$u(x) = \frac{\tan(2x)}{\sec(5x^2 + 1)} \qquad\qquad\qquad v(x) = [\sec(3x) + \cot(x^2 + 4)]^{4/7}$$

$$w(x) = [\sin^2(4x) - 7x^3]^6 \qquad\qquad\qquad x(t) = \cot^3(4t^2 + \sin^2(3t))$$

$$y(x) = \sec^{2/3}(\sqrt{x} - \tan((3x)^2 + 1)) \qquad z(x) = x|x|$$

$$\alpha(x) = \int_2^{\cos x} \sqrt{1 + t^3}\, dt \qquad\qquad\qquad \beta(x) = \int_{\sin x}^{\pi/2} t\, dt$$

5   Implicit Differentiation

Problem 5. Find the equation of the tangent line to the given curve at the point P:

1. $x^3 - xy + y^3 = 7$ and $P = (2,1)$

2. $y^4 - 4y^2 = x^4 - 9x^2$ and $P = (3,2)$

6   Intermediate Value Theorem

Problem 6. State the Intermediate Value Theorem and show that the given equation has a solution in the given Interval I:

1. $x^3 - x^2 + 2x - 7 = 0$ and $I = [1,2]$.

2. $3x - 2\tan x = 0$ and $I = [\pi/6, \pi/3]$.

3. $\cos x = x$ and $I = [0, \pi/3]$.

4. $x^2 = 2$ and $I = [1,2]$.

5. $x^7 - 5x^4 + 3x^2 - 1 = 0$ and $I = [-10, 10]$.

Make sure to state the theorem with its assumptions and conclusion, and make sure to give a well supported argument for the existence of the asserted solution of the equation.

7   Approximation by Differentials

Problem 7. Use approximation by differentials to find an approximate value of $f(x)$ given that $f(x_0) = y_0$

1. Find $f(10) = \sqrt{10}$ using that $\sqrt{9} = 3$

2. Find $f(10) = \sqrt[3]{10}$ using that $\sqrt[3]{8} = 2$

3. Find $f(24) = \sqrt{24}$ using that $\sqrt{25} = 5$

4. Find $\sin 31° = f(\frac{31\pi}{180})$ using that $\sin 30° = 1/2$

5. Find $f(5) = \sqrt{5}$ using that $\sqrt{4} = 2$

8 Related Rates

Problem 8. A light house is located on an island 5 km away from the nearest point on a straight shoreline and its light makes 6 revolutions per minute. How fast is the beam of light moving along the shoreline when it is 2 km away from P?

Problem 9. A 9 m ladder is leaning against a house when its base starts to slide away. By the time the base is 4 m from the house, the base is moving at a rate of .2 m/sec.

  (a) How fast is the top of the ladder sliding down the wall at that instant?

  (b) At which rate is the angle between the ladder and the ground changing at that instant?

Problem 10. The minute hand on a clock is 1.6 m long and the hour hand is 1 m. How fast is the distance between the tips of the hands changing at one o' clock?

Problem 11. A dinghy is pulled towards a dock by a rope from the bow through a ring on the dock 2 m above the bow. The rope is pulled in at a rate of .5 m/sec.

  (a) How fast is the boat approaching the dock when 6 m of rope are out?

  (b) How fast is the angle of incline of the rope changing at this moment?

  (c) How fast is the boat approaching the dock when the dinghy is 6 m away from the dock?

Problem 12. A street light is mounted on top of a 7 m tall pole. A man 2 m tall walks towards the pole with a speed of 1.2 m/sec. How fast does the length of shandow decrease when the man is 25 m from the light pole?

Problem 13. When thin air expands adiabatically (without gaining or losing heat, though its temperature will change), then its pressure P and volume V are related by the equation

$$PV^{1.4} = C,$$

where C is a constant. Suppose that at a certain instant the volume is .2 m³, the pressure is 80 kPa (kiloPascal; 1 Pascal equals 1 Newton per m²) and the pressure is decreasing at a rate of .5 kPa per minute. At what rate is the volume increasing at this instant?

9 Optimization Problems

Solve one of the following optimization problems.

Problem 14. Cut a string of length 5 m into two pieces. Use one piece as the perimeter of a square, and one as the perimeter of a right isoceles triangle. How long should each piece of string be so that the combined area of the square and the triangle is minimal, and how so that the area is maximal.

Problem 15. Repeat the previous problem with two of the following shapes: a circle, a right isoceles triangle, an equilateral triangle, a regular hexagon, an ellipse whose major axis is twice the minor axis.

Problem 16. Suppose that you can run 10 km per hour and swim 2 km per hour. There is a canal that is .4 km wide and very long. Find a strategy to get from one point on its shore line to another one 10 km away on the other side in the shortest amount of time.

Problem 17. Starting out with a round piece of material of radius 20 cm you bend up an edge to form a round open box of radius r and height h. Find r, h, and r/h for the box with maximal volume.

Problem 18. Starting out with a rectangular piece of cardboard of size 30 cm by 40 cm, you cut out squares at the corners, bend up the sides, and form an open box. What will the dimensions be if the volume is largest?

Problem 19. You are constructing a box. Its base is a rectangle. The ratio of width to length is 5 : 3. Which width, length, and height will the box have if you use $8000/\text{cm}^2$ of material and the box is of maximal volume?

Problem 20. Find the dimensions of a cyclinder of maximal volume inscribed into a right circular cone.

10 Discussion of Curves

Problem 21. Discuss one of the following curves on the given interval. In each case:

1. Find the x– and y– intercepts and find the intervals on which the function is positive, respectively negative.

2. Find the critical points and decide on which intervals the function increases, respectively decreases. Find the local minima and maxima.

3. Decide on which intervals the function is concave up, respectively down, and find the inflection points.

4. Find the absolute extrema on the given interval I.

5. Sketch the graphs based on the results from above.

(a) $f(x) = (x-2)^2(x+3) = x^3 - x^2 - 8x + 12$ on $I = [-2, 2]$.

(b) $g(x) = (x-1)(x-2)(x-3) = x^3 - 6x^2 + 11x - 6$ on $I = [0, 5]$.

(c) $h(x) = \dfrac{x}{x^2+1}$ on $I = [-2, 1]$.

(d) $q(x) = (x-1)(x+1)(x+2) = x^3 + 2x^2 - x - 2$ on $I = [-3, 2]$.

(e) $p(x) = (x^2 - 4)(x+1) = x^3 + x^2 - 4x - 4$ on $I = [-3, 3]$.

Problem 22. Repeat Problem 21 but include the discussion of asymptotes (horizontal, vertical, and slant).

(a) $f(x) = \dfrac{x^2 - 2x - 14}{x - 5}$

(b) $f(x) = \dfrac{x^3 - x^2 + 1}{x^2}$

(c) $f(x) = \dfrac{900}{x} + 4x$

11   Rules of Integration

Problem 23. Find five (5) of the following integrals:

$$\int \cos x \, dx$$

$$\int \tan x \sec^2 x \, dx$$

$$\int \frac{dx}{x^2 + 16}$$

$$\int x^2 \sqrt{x + 2} \, dx$$

$$\int x(x + 2)^6 \, dx$$

$$\int \frac{3x}{(x^2 + \pi)^3} \, dx$$

$$\int |x| \, dx$$

$$\int x\sqrt{x^2 + 1} \, dx$$

$$\int \frac{x}{\sqrt{4x^2 + 9}} \, dx$$

$$\int \cos^2 x \, dx$$

$$\int \tan^2 x \, dx$$

$$\int \sin 5x \, dx$$

$$\int \sec^2 x \tan^2 x \, dx$$

$$\int \cos x \sin^3 x \, dx$$

$$\int x(2x + 1)^3 \, dx$$

$$\int_{-1}^{1} \sqrt{1 - x^2} \, dx$$

$$\int t^2 (1 - t)^5 \, dt$$

$$\int \sec^2 (5x) \, dx$$

$$\int_{0}^{2} x\sqrt{x^2 + 1} \, dx$$

$$\int_{0}^{\pi/2} x \sin x^2 \, dx$$

$$\int_{-1}^{2} x|x| \, dx$$

12   Riemann Sums

Problem 24. Calculate the Riemann sum for the following situation:

(a) The function is $f(x) = \frac{1}{x}$ and the interval $I = [1, 4]$. Partition the interval using the points

$$x_0 = 1, \; x_1 = 2, \; x_2 = \frac{5}{2}, \; x_3 = \frac{7}{2}, \; x_4 = 4.$$

Use the midpoint in each of the subintervals as distinguished point.

(b) The function is $f(x) = \sin x$ and the interval $I = [0, \frac{\pi}{2}]$. Partition the interval using the points

$$x_0 = 0, \; x_1 = \frac{\pi}{6}, \; x_2 = \frac{\pi}{4}, \; x_3 = \frac{\pi}{3}, \; x_4 = \frac{\pi}{2}.$$

Use the left end point in each of the subintervals as distinguished point.

(c) Repeat (b) using right end points.

(d) The function is $f(x) = \tan x$ and the interval $I = [0, \frac{\pi}{3}]$. Partition the interval using the points

$$x_0 = 0, \; x_1 = \frac{\pi}{6}, \; x_2 = \frac{\pi}{4}, \; x_3 = \frac{\pi}{3}$$

As distinguished points use

$$x_1{}^* = \tfrac{\pi}{6},\ x_2{}^* = \tfrac{\pi}{6},\ x_3{}^* = \tfrac{\pi}{3}$$

(e) The function is $f(x) = x^2$ and the interval $I = [1, 2]$. Partition the interval into three subintervals of equal length. Use the midpoint in each of the intervals as distinguished point.

13 Areas Between Graphs and Volumes

Problem 25. Set $f(x) = 4x - x^3$ and $g(x) = x^3 - 6x^2 + 8x$.

1. Find the area of the region Ω trapped between the graphs of f and g.

2. Find the volume of the solid of revolution obtain when Ω is revolved about the y–axis.

3. Find the volume of the solid revolution when Ω is revolved about the axis $y = -3$.

Problem 26. Set $f(x) = \sin x$ and $g(x) = \dfrac{3x}{5\pi}$

1. Find the area of the region Ω trapped between the graphs of f and g in the first quadrant.

2. Find the volume of revolution obtained when Ω is revolved around the x–axis.

3. Find the volume of revolution obtained when Ω is revolved about the axis $x = -1$. (Set up the integral only.)

Problem 27. Let Ω be the region between the curves $x = y$ and $x = y(2 - y)$.

1. Find the area of Ω.

2. Find the volume of the solid of revolution when Ω is revolved around the x–axis.

3. Find the volume of the solid of revolution when Ω is revolved around the axis $x = +1$.

Problem 28. Let Ω be the region between the curves $y = 1$ and $y = x(x - 2)^2$

1. Sketch Ω and compute its area.

2. Find the volume of the solid of revolution when Ω is revolved around the x–axis.

3. Find the volume of the solid of revolution when Ω is revolved around the y–axis.