Algebra qualifying exam

University of Hawai‘i at Mānoa

Spring 2016

Part I

Instructions:

- This exam consists of 2 parts. You have 2 hours to complete each part.
- Each part has 3 sections each, for a total of 6 sections.
- Answer 6 questions in each part. Answer at least one question in each of the 6 sections; the first question in each section is generally the easiest. If you have time, you may answer more than 6 questions—your entire paper will be read and taken into consideration.

1. Group theory

1. Let S_3 be the group of permutations on 3 letters and let D_3 be the dihedral group of order 6 (i.e. the group of rotational and reflectional symmetries of an equilateral triangle). Show that $S_3 \cong D_3$.

2. Let G be a finite group.

(a) Suppose that $H \lneq G$ and $[G : H] = 2$. Show that $H \trianglelefteq G$.

(b) Suppose that $|G| = 18$. Show that G is solvable.

3. Provide counter-examples to the following false statements.

(a) Let G_1, G_2 be groups, and suppose that $H \leq G_1 \times G_2$. Then H is of the form $H_1 \times H_2$ with $H_i \leq G_i$.

(b) If $H \trianglelefteq G$ and $N \trianglelefteq H$, then $N \trianglelefteq G$.

(c) If $H_1 \trianglelefteq G_1$, $H_2 \trianglelefteq G_2$, and $G_1/H_1 \cong G_2/H_2$, then $G_1 \cong G_2$.

2. Fields and Galois theory

In this section, $\mathbf{Q}$ denotes the field of rational numbers and S_n denotes the symmetric group of degree n.

1. (a) Give an example of a specific field extension K/F that is cyclic of degree 6.

(b) Give an example of a specific field extension K/F that is Galois but not abelian.

2. Recall that a field extension K/F is called normal if it is the splitting field of some collection of polynomials $\{f_i \in F[x] : i \in I\}$. Let F be a field and let K and L be two field extensions of F contained in a common bigger extension. Suppose both K/F and L/F are normal. Show that KL/F is normal.

3. Let K/F be a separable field extension of degree 3. Let $\widetilde{K}$ be the Galois closure of K over F and suppose $\mathrm{Gal}(\widetilde{K}/F) \cong S_3$. Show that there is only one F-automorphism of K.

4. Consider the field extension $K/\mathbf{Q}$ where $K = \mathbf{Q}(\sqrt{2}, i)$ (where i is, of course, a square root of -1).

(a) Determine the minimal polynomial of $\alpha := \sqrt{2} + i$. Conclude that α is a primitive element of $K/\mathbf{Q}$.

(b) Determine all intermediate extensions of $K/\mathbf{Q}$. Justify your answer.

3. Category theory

1. (a) Let $\mathcal{C}$ be a category and let C be a fixed object of $\mathcal{C}$. For an object A of $\mathcal{C}$, let $\mathcal{F}_C(A) = \mathrm{Hom}_{\mathcal{C}}(A, C)$ be the set of morphisms in $\mathcal{C}$ from A to C. Show that $\mathcal{F}_C$ gives a contravariant functor from $\mathcal{C}$ to the category of sets; in particular, for a morphism $f : A \to B$ of $\mathcal{C}$, say what $\mathcal{F}_C(f)$ is.

(b) Suppose $\mathcal{C}$ is the category of sets and the $C = \{0, 1\}$. Show that for every set A, there is a bijection between $\mathcal{F}_C(A)$ and the set of subsets of A.

2. For a group G, let $\mathcal{C}_G$ be the associated groupoid, i.e. the category $\mathcal{C}_G$ that has exactly one element, denoted $\bullet_G$, whose morphisms are described as follows: for each $g \in G$ there is an isomorphism $f_g : \bullet_G \to \bullet_G$ and the composition is defined by $f_g \circ f_h = f_{gh}$ (i.e. $\mathrm{Hom}_{\mathcal{C}_G}(\bullet_G, \bullet_G)$ is isomorphic to G as a group).

(a) Suppose G and H are groups. Show that giving a functor $\mathcal{F} : \mathcal{C}_G \to \mathcal{C}_H$ is the same as giving a group homomorphism $\varphi : G \to H$.

(b) Given two group homomorphisms $\varphi, \psi : G \to H$, say that φ and ψ are conjugate if there is an $h \in H$ such that for all $g \in G$

$$h\varphi(g)h^{-1} = \psi(g).$$

From the previous part, given $\varphi : G \to H$, there is corresponding $\mathcal{F}_\varphi : \mathcal{C}_G \to \mathcal{C}_H$. Show that φ and ψ are conjugate if and only if there is a natural transformation $T : \mathcal{F}_\varphi \to \mathcal{G}_\psi$.

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4. Ring theory

In this section, all rings and all ring homomorphisms are unital (i.e. all rings have an identity element and all ring homomorphisms map the identity in the domain to the identity in the target). By an ideal in a ring, we mean a two-sided ideal.

1. Let $\varphi : R \to S$ be a ring homomorphism.

(a) Show that the inverse image of an ideal is an ideal.

(b) Suppose R and S are commutative. Show that the inverse image of a prime ideal is prime.

(c) Again suppose R and S are commutative. True or false:

"The inverse image of a maximal ideal is maximal."

If true, prove it; if false, provide a specific counterexample.

2. In this question, rings are commutative with identity.

(a) Give an example of a unique factorization domain (UFD) that is not a principal ideal domain (PID).

(b) Suppose R is an integral domain. Show that an element $p \in R$ is prime if and only if (p) is a (non-zero) prime ideal of R.

(c) Suppose R is an integral domain that is not a field. Show that an element $p \in R$ is irreducible if and only if (p) is a maximal among the proper principal ideals of R.

(d) Suppose R is a PID. Show that an element $p \in R$ is irreducible if and only if it is prime.

3. Let $R = \mathbf{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} : a, b \in \mathbf{Z}\}$. Define the norm function $N : R \to \mathbf{Z}_{\geq 0}$ by

$$N(a + b\sqrt{-5}) = (a + b\sqrt{-5}) \cdot (a - b\sqrt{-5}) = a^2 + 5b^2.$$

(a) Suppose $\alpha \in R$ is such that $N(\alpha)$ is a prime number. Show that α is irreducible.

(b) Show that $\sqrt{-5}$ and $3 + 2\sqrt{-5}$ are irreducible in R.

(c) Show that 2 and 3 are irreducible in R. But show that they are not prime in R.

(d) Is R a unique factorization domain?

(e) Determine $R/(\sqrt{-5})$. Is $\sqrt{-5}$ prime in R?

5. Modules and multilinear algebra

In this section, $\mathbf{R}$ denotes the field of real numbers, $\mathbf{Z}$ denotes the ring of integers, and $\mathbf{Z}/n\mathbf{Z}$ the ring of integers modulo n.

1. Simplify the following.

(a) $(\mathbf{Z}/12\mathbf{Z}) \otimes_{\mathbf{Z}} (\mathbf{Z}/16\mathbf{Z})$

(b) $\dim_{\mathbf{R}} \left(\bigwedge^3 \mathbf{R}^5 \right)$

(c) $(\mathbf{Z}/3\mathbf{Z})^2 \otimes_{\mathbf{Z}} (\mathbf{Z}/3\mathbf{Z})$

2. Let p be a prime number.

(a) Let $M = \mathbf{Z}/p\mathbf{Z}$, viewed as a $\mathbf{Z}/p\mathbf{Z}$-module. Show that M is injective.

(b) Let $N = \mathbf{Z}/p\mathbf{Z}$, viewed as a $\mathbf{Z}$ module. Show that N is not injective.

3. Give an example of each of the following.

(a) A $\mathbf{Z}$-module that is not flat.

(b) A $\mathbf{Z}/6\mathbf{Z}$-module that is projective, but not free.

(c) An exact sequence of $\mathbf{Z}$-modules that does not split.

(d) A naturally defined perfect pairing between $\bigwedge^k V$ and $\bigwedge^{n-k} V$, where V is a n-dimensional vector space.

6. Commutative algebra

In this section, $\mathbf{Q}$ denotes the field of rational numbers and $\mathbf{R}$ denotes the field of real numbers. All rings in this section are commutative with identity.

1. Give an example for each of the following.

(a) A ring of Krull dimension 0.

(b) A ring of Krull dimension 1.

(c) For a given positive integer n, a ring of Krull dimension $\geq n$.

2. Let $\mathbf{Q}[X]$ be the ring of polynomials with coefficients in $\mathbf{Q}$. Let R be the subring of $\mathbf{Q}[X]$ consisting of all polynomials whose X-coefficient is equal to 0.

(a) Show that $X^2, X^3 \in R$ are irreducible, but not prime.

(b) Show that $R = \mathbf{Q}[X^2, X^3]$, and that R is noetherian.

3. Let $\mathbf{C_0}$ be the ring of all continuous functions from $\mathbf{R}$ to $\mathbf{R}$.

(a) Show that $\mathbf{C_0}$ is not an integral domain, that it is not noetherian, and that it has no irreducible elements.

(b) What are the units in $\mathbf{C_0}$?

(c) Let $f(x) = x$. Show that the localization of $\mathbf{C_0}$ at $S = \{f^n : n \geq 0\}$ is not an integral extension of $\mathbf{C_0}$.