Sample algebra qualifying exam

University of Hawai‘i at Mānoa

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Part I

1. Group theory

In this section, D_n and C_n denote, respectively, the symmetry group of the regular n-gon (of order $2n$) and the cyclic group of order n.

1. Let $\varphi : G \to G'$ be a group homomorphism. If N' is a normal subgroup of G', show that $\varphi^{-1}(N')$ is a normal subgroup of G.

2. (a) Let $C_2 = \{1, \gamma\}$ act on C_4 with γ sending an element to its inverse. Show that $D_4 \cong C_2 \ltimes C_4$.

(b) Show that $D_6 \cong C_2 \times D_3$.

3. (a) Suppose the group G acts on the set X. Show that the stabilizers of elements in the same orbit are conjugate.

(b) Let G be a finite group and let H be a proper subgroup. Show that the union of the conjugates of H is strictly smaller than G, i.e.

$$\bigcup_{g \in G} g^{-1} H g \subsetneq G.$$

(c) Suppose G is a finite group acting transitively on a set S with at least 2 elements. Conclude from parts (a) and (b) (or otherwise), that there is an element of G with no fixed points.

2. Fields and Galois theory

In this section, $\mathbf{Q}$ denotes the field of rational numbers and $\mathbf{F}_q$ denotes the finite field of order q.

1. For each of the following, give an example and provide some justification.

(a) A separable field extension that is not normal.

(b) An inseparable field extension.

2. Consider the field $K = \mathbf{Q}(\sqrt{3}, \sqrt{7})$.

(a) Determine a primitive element for the extension $K/\mathbf{Q}$, i.e. an element $\alpha \in K$ such that $K = \mathbf{Q}(\alpha)$.

(b) Determine the minimal polynomial of the element you found in part (a).

(c) Is $K/\mathbf{Q}$ Galois? If so, what is its Galois group. If not, what is the degree of its Galois closure? Justify these answers.

3. What is the Galois group of $\mathbf{F}_{2^6}/\mathbf{F}_2$? What are the intermediate extensions of $\mathbf{F}_{2^6}/\mathbf{F}$?

4. Let K/F be a finite Galois extension with Galois group G and suppose the intermediate extensions E_1 and E_2 correspond to the subgroups H_1 and H_2 of G, respectively.

(a) Show that $E_1 \cap E_2$ corresponds to the subgroup of G generated by H_1 and H_2.

(b) Show that $H_1 \cap H_2$ corresponds to the intermediate extension E_1E_2.

3. Category theory

1. (a) Given two objects X, Y in a category, describe the setup and write out the universal property for the coproduct $X \coprod Y$.

(b) Give an example of a category where the coproduct of two objects exists, say what the coproduct is, and prove that it is the coproduct.

2. Let $\mathcal{F} : \mathbf{Grp} \to \mathbf{Set}$ be the forgetful functor sending a group to its underlying set. Let $\mathcal{G} : \mathbf{Grp} \to \mathbf{Set}$ be the functor

$$G \mapsto \mathrm{Hom}(\mathbf{Z}, G)$$

sending a group to the set of group homomorphisms from the additive group of integers to G.

(a) Show that $\mathcal{G}$ is indeed a covariant functor.

(b) Show that $\mathcal{F}$ and $\mathcal{G}$ are naturally isomorphic (i.e. show that $\mathcal{F}$ is represented by $\mathbf{Z}$).

Part II

4. Ring theory

1. For the following, give examples and provide some justification.

(a) A unique factorization domain (UFD) that is not a principal ideal domain (PID).

(b) An irreducible element a in an integral domain such that a is not a prime element.

(c) A commutative ring R with identity such that $R[x]$ has units that are not contained in R.

2. In this question, rings are commutative with identity.

(a) Show that every non-zero prime ideal in a PID is maximal.

(b) Deduce that if R is a ring such that $R[x]$ is a PID, then R is, in fact, a field.

(c) Show that, in a UFD, a non-zero element is prime if and only if it is irreducible.

3. Let R be a ring with identity, let n be a positive integer, and let $M_n(R)$ denote the ring of $n \times n$ matrices over R.

(a) Show that a subset $J \subseteq M_n(R)$ is an ideal if and only if $J = M_n(I)$, where I is an ideal of R.

(b) Now, suppose R is a division ring. Conclude that $M_n(R)$ is simple (i.e. it has no non-trivial proper ideals).

5. Modules and multilinear algebra

In this section, $\mathbf{Z}$ denotes the ring of integers and $\mathbf{Z}/n\mathbf{Z}$ the ring of integers modulo n.

1. (a) Let m and n be two positive integers. Show that

$$\mathbf{Z}/6\mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{Z}/8\mathbf{Z} \cong \mathbf{Z}/2\mathbf{Z}.$$

(b) Give an example of a $\mathbf{Z}$-module that is not flat. Justify.

(c) Give an example of a flat $\mathbf{Z}$-module that is not projective. Justify.

2. Let R be a ring and suppose that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & {\scriptstyle\alpha}\big\downarrow & & {\scriptstyle\beta}\big\downarrow & & {\scriptstyle\gamma}\big\downarrow & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

is a commutative diagram of R-modules whose rows are short exact sequences. Show that if α and γ are isomorphisms, then β is also an isomorphism.

3. Let V be a finite-dimensional vector space.

(a) Suppose $\{e_1, \ldots, e_d\}$ is a basis of V. Show that

$$\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n} : (i_1, \ldots, i_n) \text{ varying over all } n\text{-tuples such that } 1 \le i_j < i_{j+1} \le d\},$$

is a basis of $\bigwedge^n V$ for $n \le d$.

(b) Let $\{v_1, \ldots, v_n\}$ be a set of vectors in V. Show that they are linearly independent if and only if $v_1 \wedge \cdots \wedge v_n \neq 0$ in $\bigwedge^n V$.

(c) Suppose V is three-dimensional. Provide a natural perfect pairing between V and $\bigwedge^2 V$.

6. Commutative algebra

In this section, all rings are commutative (with identity), all ring homomorphisms and all modules are unital.

1. Let $\mathbf{C}$ denote the field of complex numbers. What are the prime ideals in $\mathbf{C}[x]$? Describe the localization of $\mathbf{C}[x]$ at these prime ideals.

2. Let A be a subring of B with B integral over A.

(a) Suppose $a \in A$ is a unit in B, then show it is a unit in A.

(b) Suppose A and B are both integral domains. Show that A is a field if and only if B is a field.