ANALYSIS QUALIFYING EXAM - SEPTEMBER 2016

Attempt the following six problems. Please note the following:

- Throughout this exam, unless otherwise indicated, integration is with respect to Lebesgue measure.
- We denote the Lebesgue measure of a set A by $m(A)$.
- Partial credit will be given for partially correct solutions, even if incomplete.
- The parts of problems are not equally difficult, and will not be weighted equally.
- Good luck!

(1) For each n, let $f_n : [0,1] \to \mathbb{R}$ be a continuous function, and assume that the sequence (f_n) converges pointwise everywhere on $[0,1]$ to a limit function $f : [0,1] \to \mathbb{R}$.

(a) Give an example where the limit function is not continuous.

(b) Show from the definitions that the limit function is a Borel function.

(2) In this problem, let $0 \cdot \infty = 0$, and let $f : \mathbb{R}^d \to [0,\infty)$ be a non-negative, simple function.

(a) Show that

$$\int_{\mathbb{R}^d} f(x)\mathrm{d}x = \sup \left\{ \sum_{k=1}^{N} \inf\{f(x) \mid x \in E_k\}\, m(E_k) \right\}$$

where the supremum is taken over all finite, disjoint families of measurable sets $E_1, ..., E_N$.

(b) Show that the equality still holds when the sets E_k are only allowed to be closed sets.

(c) Show that the equality does not hold when $d = 1$ and the sets E_k are only allowed to be open sets. You may use standard facts about the existence of subsets of $\mathbb{R}$ with certain properties as long as you state them clearly.

(3) Let (a_n) be a decreasing sequence of positive real numbers so that

$$\lim_{n\to\infty} a_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} a_n \log(\frac{1}{a_n}) < \infty.$$

Suppose (b_n) is an arbitrary sequence of real numbers. Let

$$f_n(x) = \begin{cases} \frac{a_n}{|x-b_n|} & a_n \leqslant |x-b_n| \\ 0 & \text{otherwise} \end{cases}$$

and set $A_n = \{x \mid f_n(x) = 0\} = (b_n - a_n, b_n + a_n)$.

(a) Show that $\sum_{n=1}^{\infty} m(A_n) < \infty$.
(b) Show that $m(\bigcap_{k=1}^{\infty} \bigcup_{n\geqslant k} A_n) = 0$.
(c) For every $R > 0$ show that $\int_{-R}^{R} \sum_{n=1}^{\infty} f_n(x) \mathrm{d}x < \infty$
(d) Show that $\sum_{n=1}^{\infty} \frac{a_n}{|x-b_n|}$ converges for almost every x in $\mathbb{R}$.

(4) For $f, g \in L^1(\mathbb{R}^d)$, let the convolution $f * g$ be defined as

$$f * g(x) = \int_{\mathbb{R}^d} f(x-t)g(t)\mathrm{d}t.$$

Prove the following:

(a) $f * g \in L^1(\mathbb{R}^d)$. You may use without proof that the function $\mathbb{R}^{2d} \to \mathbb{C}$ defined by $(x,t) \mapsto f(x-t)g(t)$ is measurable.
(b) If g is bounded, then $f * g$ is continuous.

(5) Let $p > q$ be fixed numbers in $[1, \infty)$. Let $C_b(\mathbb{R})$ denote the continuous, bounded, complex-valued, functions on $\mathbb{R}$. Which of the two inclusions

$$C_b(\mathbb{R}) \cap L^p(\mathbb{R}) \subseteq L^q(\mathbb{R}) \quad \text{and} \quad C_b(\mathbb{R}) \cap L^q(\mathbb{R}) \subseteq L^p(\mathbb{R})$$

is correct, and which is not? Prove the correct one, and give a counter-example to the false one.

(6) A function $f : [0,1] \to \mathbb{R}$ is *Lipschitz* if there is a constant $c \geqslant 0$ such that

$$|f(x) - f(y)| \leqslant c|x - y|$$

for all $x, y \in [0,1]$.

(a) Give an example of a Lipschitz function $f : [0,1] \to \mathbb{R}$ that is not differentiable at infinitely many points in $[0,1]$.
Partial credit for a good picture of the graph of an appropriate f, as long as it's clearly explained.

(b) Show that if $f : [0,1] \to \mathbb{R}$ is Lipschitz, then there is at least one point in $(0,1)$ where f is differentiable.