ANALYSIS QUALIFYING EXAM - October 2017

Attempt the following six problems. Please note the following:

- Throughout this exam, unless otherwise indicated, integration is with respect to Lebesgue measure, and $L^p(E)$ denotes the Lebesgue space of a subset E of $\mathbb{R}^d$ with respect to Lebesgue measure.
- Partial credit will be given for partially correct solutions, even if incomplete.
- The parts of problems are not equally difficult, and will not be weighted equally.
- Good luck!

(1) (a) Let B denote the unit ball $\{x \in \mathbb{R}^d \mid |x| \leqslant 1\}$ in $\mathbb{R}^d$. Give an example of a sequence (f_n) of continuous functions in $L^1(B)$ such that $\|f_n\|_1 \to 0$ but such that (f_n) does not tend to zero pointwise almost everywhere.

(b) Show that such an example does not exist if the sequence is additionally assumed to be equicontinuous (recall this means the for all $\epsilon > 0$ there exists $\delta > 0$ such that for all n, if $|x - y| < \delta$ then $|f_n(x) - f_n(y)| < \epsilon$).

(2) Say (f_n) is a sequence in $L^1[0,1]$ converging to some $f \in L^1[0,1]$ pointwise almost everywhere. Assume moreover that $\|f_n\|_1 \to \|f\|_1$ as $n \to \infty$, and show that $f_n \to f$ in $L^1[0,1]$.

(3) Let $f : \mathbb{R} \to \mathbb{R}$ be integrable. Assume that there exists $\epsilon > 0$ such that $|\int_A f| \leqslant m(A)^{1+\epsilon}$ for all measurable sets $A \subseteq \mathbb{R}$. Show that f equals zero almost everywhere.

(4) Fix $p, q \in [1, \infty)$. Let X be the vector space of measurable functions $f : [0, \infty) \to \mathbb{R}$ such that the norm defined by

$$\|f\| := \|f|_{[0,1]}\|_p + \|f|_{[1,\infty)}\|_q$$

is finite, where $\|\cdot\|_p$ and $\|\cdot\|_q$ are the norms on $L^p[0,1]$ and $L^q[1,\infty)$ respectively and $f|_S$ denotes the restriction of f to a set S.

(a) Show that $(X, \|\cdot\|)$ is complete.
You may use without proof that standard L^p spaces are complete.

(b) Under what conditions (if any) on p and q is X contained in $L^p[0, \infty)$, and under what conditions is it contained in $L^q[0, \infty)$? Justify your answers.

(5) For a function $f : \mathbb{R} \to \mathbb{R}$ and $h \in \mathbb{R}$, let $\tau_h f$ be defined by $\tau_h f(x) = f(x + h)$.

(a) Give an example of an integrable function $f : \mathbb{R} \to \mathbb{R}$ such that $\lim_{h\to 0} \tau_h f(x)$ does not exist for any $x \in \mathbb{R}$.

(b) Show that for any $f \in L^1(\mathbb{R})$, $\lim_{h\to 0} \|\tau_h f - f\|_1 = 0$.

(6) For measurable functions $f, g : \mathbb{R} \to \mathbb{C}$, provisionally define the convolution $f * g$ by

$$(f * g)(x) := \int_{\mathbb{R}} f(y)g(x - y)dy.$$

You may assume without proof that the integrand above is measurable.

(a) Show that if $f \in L^1(\mathbb{R})$ and $g \in L^\infty(\mathbb{R})$, then $f * g \in L^\infty(\mathbb{R})$ and $\|f * g\|_\infty \leqslant \|f\|_1 \|g\|_\infty$.

(b) Show that if $f \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R})$ for $p \in [1, \infty)$, then $f * g \in L^p(\mathbb{R})$ and $\|f * g\|_p \leqslant \|f\|_1 \|g\|_p$.