

ANALYSIS QUALIFYING EXAM - AUGUST 2018

Attempt the following six problems. Please note the following:

- Throughout this exam, $L_p(E)$ denotes the Lebesgue space of a measure space E . Measure and integration on \mathbb{R}^d use Lebesgue measure unless otherwise stated.
- Partial credit will be given for partially correct solutions, even if incomplete.
- The parts of problems are not equally difficult, and will not be weighted equally.
- Good luck!

- (1) Suppose (E, Σ, μ) is a measure space and $\|g\|_{L_1(E)} \neq 0$. Prove that if $f : E \rightarrow \mathbb{C}$ is Σ measurable, then

$$\frac{|\int_E f(x) d\mu(x)|^2}{\int_E |g(x)| d\mu(x)} \leq \int_E \frac{|f(x)|^2}{|g(x)|} d\mu(x).$$

- (2) Compute the limit

$$\lim_{n \rightarrow \infty} \int_0^\infty \frac{n}{x(1+x^2)} \sin\left(\frac{x}{n}\right) dx.$$

Justify your answer.

- (3) Let $f \in L_2[0, 1]$ be a function such that $\int_0^1 x^n f(x) dx = 0$ for all integers $n \geq 0$. Show that f is zero almost everywhere.

- (4) Let (r_n) be an enumeration of the rationals and set $A_n = (r_n, r_n + n^{-3})$. Let

$$f = \sum_{n=1}^{\infty} n \chi_{A_n}.$$

- (a) Show that f is finite almost everywhere.
- (b) Show that $f \notin L_2(\mathbb{R})$.
- (c) Is f in $L_2[0, 1]$? There are three possible answers: “yes”, “no”, or “depends on the given enumeration of the rationals”. Say which holds, and justify your answer.
- (5) Let (X, Σ, μ) be a finite measure space, \mathcal{Y} a sub- σ -algebra of Σ , and $\nu = \mu|_{\mathcal{Y}}$. If $f \in L_1(\mu)$, show that there is $g \in L_1(\nu)$ so that $\int_Y f d\mu = \int_Y g d\nu$ for all $Y \in \mathcal{Y}$. Make sure you justify why the hypotheses of any theorem you use are satisfied. Also, explain why it may not suffice to simply take $g = f$.

- (6) For a Lebesgue measurable subset $A \subset [0, 1] \times [0, 1]$, define the slice at x by

$$A_1(x) = \{y \in [0, 1] \mid (x, y) \in A\}$$

and the slice at y by

$$A_2(y) = \{x \in [0, 1] \mid (x, y) \in A\}.$$

- (a) Let μ be Lebesgue measure on $[0, 1]$. Explain why

$$\int_0^1 \mu(A_1(x)) d\mu(x) = \int_0^1 \mu(A_2(y)) d\mu(y).$$

- (b) Let ν be the counting measure on $[0, 1]$ (i.e., ν is defined on the σ -algebra of all subsets of $[0, 1]$ by setting $\nu(E)$ to be the cardinality of E if this is finite, and ∞ if E is infinite). Does the identity

$$\int_0^1 \mu(A_1(x)) d\nu(x) = \int_0^1 \nu(A_2(y)) d\mu(y)$$

necessarily hold? Either prove it or provide a counter-example.