ANALYSIS QUALIFYING EXAM - AUGUST 2019

Attempt the following six problems. Please note the following:

- Throughout this exam, unless otherwise indicated, m denotes the Lebesgue measure in $\mathbb{R}^d$, integration is with respect to m, and $L_p(E)$ denotes the Lebesgue space of a subset E of $\mathbb{R}^d$ with respect to m.

- Partial credit will be given for partially correct solutions, even if incomplete.

- The parts of problems are not equally difficult, and will not be weighted equally.

- Good luck!

(1) Let (X, Σ, μ) be a measure space, and let $\{A_j \mid j \in \mathbb{N}\} \subset \Sigma$ be a countable family of measurable sets. For $n \in \mathbb{N}$, define

$$B_n := \{x \in \mathbb{R}^d \mid \#\{j \in \mathbb{N} \mid x \in A_j\} \geq n\}$$

(i.e., B_n is the set of points appearing in at least n of the A_js). Show that

(a) $B_n \in \Sigma$.

(b) $\mu(B_n) \leq \frac{1}{n} \sum_{j=1}^{\infty} \mu(A_j)$.

(2) Let $BV(\mathbb{R})$ denote the space of functions of bounded variation on $\mathbb{R}$, and for $f : \mathbb{R} \to \mathbb{R}$, define the *total variation* as

$$\mathrm{TV}(f) := \sup\left\{ \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| \;\middle|\; x_0 < x_1 \cdots < x_n \right\}.$$

Let $C_c^1(\mathbb{R})$ denote the space of continuously differentiable, compactly supported functions on $\mathbb{R}$. Do the following:

(a) Provide a function in $BV(\mathbb{R})$ which is not in $C_c^1(\mathbb{R})$ and compute its total variation.

(b) Show that for $f \in C_c^1(\mathbb{R})$, $\mathrm{TV}(f) \leq \|f'\|_{L_1(\mathbb{R})}$.

(c) Show that for $f \in C_c^1(\mathbb{R})$, $\|f'\|_{L_1(\mathbb{R})} \leq \mathrm{TV}(f)$.

(3) Let f and g be Lebesgue integrable real valued functions on $[0,1]$ and define

$$F(x) = \int_0^x f(t)\,dt, \quad \text{and} \quad G(x) = \int_0^x g(t)\,dt.$$

Use Fubini's Theorem to verify the integration by parts formula

$$\int_0^1 F(x)g(x)\,dx = F(1)G(1) - \int_0^1 f(x)G(x)\,dx.$$

Justify that the hypotheses of Fubini's Theorem are satisfied.

(4) Suppose that $\{f_n\}$ is a sequence of Lebesgue measurable functions on $[0,1]$ such that $\lim_{n\to\infty} \int_0^1 |f_n(t)|\,dt = 0$ and that there is an integrable function g on $[0,1]$ such that $|f_n|^2 \leq g$, for all n. Do the following:

(a) Give an example showing that $\{f_n\}$ does not necessarily converge a.e.

(b) Prove that $\lim_{n\to\infty} \int_0^1 |f_n(t)|^2\,dt = 0$.

(5) Suppose that $f \in L^1[0,1]$ satisfies $\int_E f(t)\,dt = 0$ for all measurable sets $E \subset [0,1]$ of Lebesgue measure $2/3$. Prove that $f = 0$ a.e.

(6) Let $e_k \in \ell_2(\mathbb{N})$ denote the sequence $e_k(j) = \delta_{j,k}$ (i.e., it's the sequence which is zero except for a single 1 in the kth entry). Let $\ell_0(\mathbb{N})$ be the vector space of *finitely* supported sequences

$$\ell_0(\mathbb{N}) = \left\{ \sum_{k=1}^{\infty} a_k e_k \mid \#\{j \mid a_j \neq 0\} < \infty \right\}.$$

(a) Show that the map $\lambda : \ell_0(\mathbb{N}) \to \mathbb{C} : \sum_{j=1}^{N} a_j e_j \mapsto \sum_{j=1}^{N} a_j$ does not satisfy the following property:

$$(\exists f \in \ell_2(\mathbb{N}))(\forall \tau \in \ell_0(\mathbb{N})) \quad \lambda(\tau) = \sum_{j=1}^{\infty} \tau(j)\overline{f(j)}.$$

(b) Give the product $\ell_2(\mathbb{N}) \times \mathbb{C}$ the norm $\|\cdot\| : (x,c) \mapsto \sqrt{\|x\|_{\ell_2}^2 + |c|^2}$, and let

$$G(\lambda) := \{(\tau, a) \in \ell_2(\mathbb{N}) \times \mathbb{C} \mid \tau \in \ell_0(\mathbb{N}), a = \lambda\tau\}$$

denote the graph of λ in $\ell_2(\mathbb{N}) \times \mathbb{C}$. Show that the closure $\overline{G(\lambda)}$ fails the vertical line test: i.e., there are points $(x,a), (x,b) \in \overline{G(\lambda)}$ with $a \neq b$.
Hint: consider $x = \sum_{j=1}^{\infty} \frac{(-1)^j}{j} e_j$