ANALYSIS QUALIFYING EXAM, FALL 2020

- Throughout this exam, a measure space denotes a triple $(X, \mathcal{S}, \mu)$ where X is a set, $\mathcal{S}$ is a σ-algebra on X, and μ is a (positive) measure on $(X, \mathcal{S})$.
- Partial credit will be given for partially correct solutions, even if incomplete.
- The parts of problems are not equally difficult, and will not be weighted equally.
- Good luck!

Problem 1

Let $(X, \mathcal{S}, \mu)$ be a measure space, and let $f : X \to [-\infty, \infty]$ be measurable.

a. Show that for every $0 < M < \infty$, we have

$$\mu(\{x \in X : |f(x)| \geq M\}) \leq \frac{1}{M} \int_X |f| \, d\mu.$$

b. Let $f_n : X \to \mathbb{R}$ be a sequence of measurable functions on $(X, \mathcal{S}, \mu)$. We say that (f_n) converges *in measure* to a measurable function $f : X \to \mathbb{R}$ if for every $\epsilon > 0$,

$$\mu(\{x \in X : |f_n(x) - f(x)| \geq \epsilon\}) \to 0 \qquad (n \to \infty).$$

Show that if $\|f_n - f\|_1 \to 0$ as $n \to \infty$, then $f_n \to f$ in measure.

Problem 2

Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := x^2$, and let $\mathcal{S} := \{A \in \mathcal{B}(\mathbb{R}) : A = -A\}$. Here $\mathcal{B}(\mathbb{R})$ is the σ-algebra of Borel subsets of $\mathbb{R}$, and $-A$ denotes the set $\{-a : a \in A\}$.

a. Show that $\mathcal{S}$ is a σ-algebra on $\mathbb{R}$.

b. Show that f is $\mathcal{S}$-measurable.

c. Show that if $\mathcal{S}'$ is another σ-algebra on $\mathbb{R}$ for which f is $\mathcal{S}'$-measurable, then $\mathcal{S} \subset \mathcal{S}'$.

Problem 3

Let $a := (a_n)_{n\geq 1}$ be a sequence of numbers in $(0, 1/2)$. We let $C_0 := [0, 1]$, and construct C_n for $n \geq 1$ inductively as follows:

Suppose that C_{n-1} is a union of disjoint closed intervals each of length l. Then we construct C_n by removing, in the middle of each interval of C_{n-1}, an open interval of length $l(1 - 2a_n)$. We define $C = C(a) := \cap_{n\geq 1} C_n$, the *Cantor set* corresponding to the sequence a.

a. Explain why C is compact and non-empty.

b. Show that C does not contain any open interval.

c. What is the Lebesgue measure of C in terms of the sequence a?

Problem 4

Let $(X, \mathcal{S}, \mu)$ be a measure space, and let $f, f_n : X \to [0, \infty]$ be measurable functions such that

$$\lim_{n\to\infty} f_n(x) = f(x) \qquad (x \in X)$$

and

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

Show that if $\int_X f \, d\mu < \infty$, then for each $E \in \mathcal{S}$, we have

$$\lim_{n\to\infty} \int_E f_n \, d\mu = \int_E f \, d\mu.$$

Problem 5

Consider the function $f(x, y) := 2e^{-2xy} - e^{-xy}$ defined on $[0, \infty) \times [0, 1]$.

a. Show that

$$\int_0^1 \int_0^\infty f(x, y) \, dx \, dy = 0.$$

b. Show that

$$\int_0^\infty \int_0^1 f(x, y) \, dy \, dx = \log 2.$$

Hint: Use the identity

$$\frac{e^{-x} - e^{-2x}}{x} = \int_1^2 e^{-xy} \, dy.$$

c. Is $|f|$ integrable on $[0,\infty)\times[0,1]$? Explain.

Problem 6

Show that there is no measurable set $E \subset \mathbb{R}$ that satisfies

$$\frac{1}{100} < \frac{m(E \cap I)}{m(I)} < \frac{99}{100}$$

for every interval $I \subset \mathbb{R}$, where m denotes the Lebesgue measure on $\mathbb{R}$.