ANALYSIS QUALIFYING EXAM - AUGUST 2021

Attempt the following problems. Please note the following:

- Throughout this exam, unless otherwise indicated, m denotes the Lebesgue measure in $\mathbb{R}^d$, integration is with respect to m, and $L_p(E)$ denotes the Lebesgue space of a subset E of $\mathbb{R}^d$ with respect to m and $p \in [1, \infty]$.
- Partial credit will be given for partially correct solutions, even if incomplete.
- The parts of problems are not equally difficult, and will not be weighted equally.
- Good luck!

(1) For the following statements, say if they are true or false, and give a proof or counterexample as appropriate.

(a) An open subset of $[0, 1]$ which is dense has Lebesgue measure one.

(b) An open subset of $[0, 1]$ which has Lebesgue measure one is dense.

(2) Let (f_n) be a sequence of continuous functions from $(0, 1)$ to $\mathbb{R}$. Say (f_n) satisfies *condition (D)* if the following two conditions are satisfied:

(D_1) (f_n) converges uniformly to a function $f : (0, 1) \to \mathbb{R}$;

(D_2) each f_n is differentiable with $|f_n'(x)| \leq 1$ for all $x \in (0, 1)$.

(a) Give an example of a sequence satisfying condition (D) such that the limit function f is not differentiable.

(b) Show that for any sequence (f_n) satisfying condition (D), the limit function f is absolutely continuous.

(3) Let $\mathcal{B}$ denote the σ-algebra of Borel subsets of $\mathbb{R}$, and let $\nu : \mathcal{B} \to [0, \infty)$ be a measure (note that $\nu(\mathbb{R}) < \infty$). For $t \in \mathbb{R}$ define

$$f(t) := \int_{\mathbb{R}} \cos(tx) \mathrm{d}\nu(x).$$

(a) Show that f is well-defined, bounded and continuous.

(b) Prove the following version of the Riemann-Lebesgue lemma: if ν as above is absolutely continuous with respect to Lebesgue measure, then $\lim\limits_{t\to\infty} f(t) = 0$.

(c) Give an example of a measure $\nu : \mathcal{B} \to [0, \infty)$ for which $\lim\limits_{t\to\infty} f(t) \neq 0$.

(4) (a) Suppose $1 \leq p \leq q \leq \infty$. Show that if $f \in L_p(\mathbb{R}) \cap L_q(\mathbb{R})$ then $f \in L_r(\mathbb{R})$ for all $r \in [p, q]$.

(b) Determine whether the following statement is true.

"If f is integrable and continuous then $f \in L_p(\mathbb{R})$ for some $p \in (1, \infty]$."

Support your answer with a proof or a counterexample.

(5) Suppose $g : (0, \infty) \to \mathbb{R}$ is a nonzero, continuous function. Show that $G : [0, 1] \times [1, \infty) \to \mathbb{R}$ defined by

$$G(x, y) = g(xy)$$

is not in $L_1\big([0, 1] \times [1, \infty)\big)$.

(6) Show that for any positive real numbers r and s,

$$\int_0^1 \frac{x^{r-1}}{1 + x^s} = \sum_{n=0}^{\infty} \frac{(-1)^n}{r + ns}.$$

Make sure you properly cite any convergence results used in your response.