ANALYSIS QUALIFYING EXAM, FALL 2022

- Throughout this exam (X, μ) denotes an arbitrary measure space, and $L^p(\mu)$ the corresponding Lebesgue space. Also, m denotes the Lebesgue measure in $\mathbb{R}^d$, and $L^p(E)$ denotes the Lebesgue space (real-valued functions) of a subset E of $\mathbb{R}^d$ with respect to m.
- Partial credit will be given for partially correct solutions, even if incomplete.
- The parts of problems are not equally difficult, and will not be weighted equally.
- Good luck!

Problem 1. Give a proof or a counterexample to each of the following statements.

a. Any subset of $[0, 1]$ with measure one is dense.

b. Any subset of $[0, 1]$ with measure one contains an open interval.

c. If $E, F \subseteq [0, 1]$ both have measure one, then $E \times F \subseteq [0, 1] \times [0, 1]$ also has measure one.

d. If $E \subseteq [0, 1] \times [0, 1]$ has measure one, then the "slice" $E_0 := \{x \in [0, 1] \mid (x, 0) \in E\}$ has measure one.

e. There exists an open, dense subset of $[0, 1]$ with measure less than $1/2$.

Problem 2. Let (X, μ) be a finite measure space. Show that

$$\lim_{n\to\infty} \int_X \frac{|f_n - f|}{1 + |f_n - f|} \, d\mu = 0$$

if and only if $f_n \to f$ in measure as $n \to \infty$.

Problem 3. For a (real-valued) function f on $\mathbb{R}$ define

$$f^y(x) = f(x-y), \qquad y \in \mathbb{R}.$$

a. Show that if $f \in L^p(\mathbb{R})$ for some $1 \leq p < \infty$ then $\|f^y - f\|_p \to 0$ as $y \to 0$.

b. Give an example of an $f \in L^\infty(\mathbb{R})$ such that $\|f^y - f\|_\infty \not\to 0$ as $y \to 0$.

Problem 4.

a. A version of Egorov's theorem states that if (X, μ) is a finite measure space and $(f_n : X \to \mathbb{R})$ is a sequence of measurable functions that converge pointwise to a function $f : X \to \mathbb{R}$, then for any $\epsilon > 0$ there exists a measurable subset E of X such that $\mu(X \setminus E) < \epsilon$ and (f_n) converges uniformly to f on E.

Is this true without the hypothesis that $\mu(X) < \infty$? Deduce the version where X has infinite measure from the version above, or give a counterexample.

b. A version of Luzin's theorem states that if $f : [a, b] \to \mathbb{R}$ is a measurable function for some $a, b \in \mathbb{R}$, then for any $\epsilon > 0$ there exists a closed subset F of $[a, b]$ such that $\mu([a, b] \backslash F) < \epsilon$ and the restriction $f|_F : F \to \mathbb{R}$ is continuous.

Is this true if we replace $[a, b]$ by $\mathbb{R}$? Deduce the version with $[a, b]$ replaced by $\mathbb{R}$ from the version above, or give a counterexample.

Problem 5. Let (X, μ) be a σ-finite measure space, and let $K(x, y)$ be a function of two variables that is measurable with respect to the product σ-algebra. Assume there is a constant C such that

$$\int_X |K(x,y)| d\mu(y) \leq C, \qquad \text{for a.e. } x \in X$$

and

$$\int_X |K(x,y)| d\mu(x) \leq C, \qquad \text{for a.e. } y \in X.$$

For $1 \leq p \leq \infty$ define

$$Tf(x) = \int_X K(x,y) f(y) d\mu(y).$$

Show that $\|T(f)\|_p \leq C\|f\|_p$. Make sure you state explicitly where the σ-finiteness assumption is used.

Problem 6.

a. Let (X, μ) be $\mathbb{N}$ with counting measure. Show that if $1 \leq p \leq q \leq \infty$, then $L^p(X) \subseteq L^q(X)$.

b. Let (X, μ) be a finite measure space with Lebesgue measure. Show that if $1 \leq p \leq q \leq \infty$, then $L^q(X) \subseteq L^p(X)$.

c. Let $X = [0,1] \times [0,1]$, equipped with Lebesgue measure. Give an example of a function $f \in L^1(X)$ that is not in $L^p(X)$ for any $p > 1$.
Note: your function f should work for all p at once! It is not enough to prove "for all p there exists $f \in L^1(X)$ that is not in $L^p(X)$".