ANALYSIS QUALIFYING EXAM - AUGUST 2023

Attempt the following six problems. Please note the following:

- Throughout this exam, $L^p(X)$ denotes the L^p-space of a measure space $(X, \mathcal{M}, \mu)$ with the associated norm of a function being $\|f\|_p$. Subsets of $\mathbb{R}^d$ are equipped with the Lebesgue sigma algebra and Lebesgue measure m_d unless otherwise stated.

- Partial credit will be given for partially correct solutions, even if incomplete.

- The parts of problems are not equally difficult, and will not be weighted equally.

- Good luck!

(1) Prove the following statement, or give a (justified) counterexample: "for every Borel measurable function $f : [0, 1] \to \mathbb{R}$ there is a continuous function $g : [0, 1] \to \mathbb{R}$ such that f equals g almost everywhere".

(2) Let $\{ A_n \}$ be a sequence of disjoint measurable subsets of $[0, 1]$ whose union is $[0, 1]$, and let B_n be a sequence of measurable subsets of $[0, 1]$ such that
$$\lim_{n \to \infty} m(B_n \cap A_k) = 0$$
for each k. Prove that
$$\lim_{n \to \infty} m(B_n) = 0.$$

(3) For this problem, assume that $\mathbb{N}$ *includes* 0.
  (a) Let $f \in L^2[-1, 1]$, and for $n \in \mathbb{N}$, let $p_n(x) = x^n$. Show that if $\int_{[-1,1]} f p_n = 0$ for all n, then $f = 0$ almost everywhere.

  (b) Give an example of $f \in L^2[-1, 1]$ such that $\int_{[-1,1]} f p_n = 0$ for all $n \in 2\mathbb{N} = \{2n \mid n \in \mathbb{N}\}$, but f is not equal to zero almost everywhere. Justify your answer.

(4) Let $(a_{n,m})_{n,m \in \mathbb{N}}$ be a collection of real numbers, indexed by pairs $(n, m) \in \mathbb{N}^2$ of non-negative integers.
  (a) *Using a result from integration theory* justify the fact that if $a_{n,m} \geq 0$ for all n, m, we have that
  $$\sum_{n \in \mathbb{N}} \left(\sum_{m \in \mathbb{N}} a_{n,m} \right) = \sum_{m \in \mathbb{N}} \left(\sum_{n \in \mathbb{N}} a_{n,m} \right)$$
  (this includes the case that both sides are equal to infinity).

  (b) Give an example of a collection $(a_{n,m})_{n,m \in \mathbb{N}}$ of real numbers such that both
  $$\sum_{n \in \mathbb{N}} \left(\sum_{m \in \mathbb{N}} a_{n,m} \right) \quad \text{and} \quad \sum_{m \in \mathbb{N}} \left(\sum_{n \in \mathbb{N}} a_{n,m} \right)$$
  converge, but the two are not equal.
  Hint: there is an example such that $a_{n,m} = 0$ unless $n = m$ or $n = m + 1$.

  (c) Explain why your solution to the previous problem does not contradict Fubini's theorem.

(5) Suppose $A \subset \mathbb{R}$ satisfies $m_1(A) = 0$, where m_1 denotes the usual Lebesgue measure on $\mathbb{R}$. Suppose $f : \mathbb{R} \to \mathbb{R}^2$ satisfies $|f(x) - f(y)| \leq \sqrt{|x - y|}$ for every x, $y \in \mathbb{R}$ (here $|\cdot|$ denotes the usual Euclidean norm on $\mathbb{R}^2$ or $\mathbb{R}$ as appropriate). Show that $m_2(f(A)) = 0$, where m_2 denotes the usual Lebesgue measure on $\mathbb{R}^2$.

(6) Let $(X, \mathcal{M}, \mu)$ be a measure space and let f be a measurable function on X. Prove that the (possibly infinite) limit

$$\lim_{n \to \infty} \int |f|^{1/n} \, d\mu$$

exists and find it in terms of f.