

## QUALIFYING EXAM - APRIL 2016

Attempt the following six problems. Please note the following:

- Throughout the exam, unless indicated otherwise, integration is with respect to Lebesgue measure.
- We denote the Lebesgue measure of a set  $A$  by  $m(A)$ .

(1) Let  $X$  be a nonempty set, let  $\mathcal{P}(X)$  denote its power set and let  $\mu : \mathcal{P}(X) \rightarrow [0, \infty]$  be the “counting measure” (i.e.,  $\mu(A) = \#A$  if  $A \subset X$  is a finite set, and  $\mu(A) = \infty$  otherwise). It is known that  $(X, \mathcal{P}(X), \mu)$  is a measure space (you don’t have to show this). Suppose  $\int_X |f(x)| d\mu(x) < \infty$ . Show that  $f$  has countable support.

(2) For  $f \in C([0, 1])$ , prove that

$$\lim_{n \rightarrow \infty} n \int_0^1 e^{-nx} f(x) dx$$

exists and find the limit.

(3) In this problem, let a rectangle in  $\mathbb{R}^2$  be a set of the form  $[a, b] \times [c, d]$ . For a subset  $E \subset \mathbb{R}^2$  and  $t \in \mathbb{R}^2$ , define  $E + t = \{e + t \mid e \in E\}$ .

(a) Consider the statement

For any measurable subset  $E \subset \mathbb{R}^2$  and for any  $\epsilon > 0$  there exists a finite union of rectangles  $Q = \bigcup_{k=1}^N ([a_k, b_k] \times [c_k, d_k])$  so that

- $E \subset Q$
- $m(Q \setminus E) < \epsilon$ .

Give a simple proof (if true) or a simple counterexample (if false).

(b) Suppose  $E$  is a measurable subset of  $\mathbb{R}^2$  having finite measure. Show that if  $\lim_{t \rightarrow 0} m(E \cap (E + t)) = 0$  then  $m(E) = 0$ .

- (4) In this problem, let  $B_R$  denote the ball of radius  $R$  in  $\mathbb{R}^2$ :  $B_R = \{x \mid |x| < R\}$ . Let  $f$  and  $g$  be measurable functions on  $\mathbb{R}^2$  satisfying the following: there exists a constant  $C > 0$  so that for every  $r > 0$ ,

$$\int_{B_{2r} \setminus B_r} |f(x)|^3 dx < Cr^1 \quad \text{and} \quad \int_{B_{2r} \setminus B_r} |g(x)|^4 dx < Cr^{-7}$$

- (a) Is  $f \in L_1(B_1)$ ? Either prove it, or provide a counterexample.
- (b) Is  $g \in L_1(B_1)$ ? Either prove it, or provide a counterexample.
- (c) Suppose, in addition to the above hypotheses, that  $f$  and  $g$  are continuous. Show that  $fg \in L_1(\mathbb{R}^2)$ .
- (5) Let  $(f_n)$  be a sequence of integrable functions satisfying

$$\int_{\mathbb{R}} |f_n(x)| dx \leq 2^{-n}$$

for all  $n \in \mathbb{N}$ . Show that  $\lim_{n \rightarrow \infty} f_n(x) = 0$  almost everywhere.

- (6) Let  $F : \mathbb{R} \rightarrow \mathbb{C}$  be an integrable function for which there exists a compact set  $K \subset \mathbb{R}$  so that  $F(x) = 0$  for almost every  $x \in \mathbb{R} \setminus K$ . For a continuous function  $\phi$ , define the convolution  $F * \phi$  as  $F * \phi(x) = \int_{\mathbb{R}} F(x - y)\phi(y)dy$ .
- (a) Prove that if  $\phi$  is absolutely continuous, then  $\frac{d}{dx}(F * \phi) = F * \phi'$  almost everywhere. *Hint: Prove it first for suitably nice  $\phi$ .*
- (b) Show that if  $\phi$  is a polynomial, then  $F * \phi$  is a polynomial as well.