ANALYSIS QUALIFYING EXAM - APRIL 2017

Attempt the following six problems. Please note the following:

- Throughout this exam, unless otherwise indicated, integration is with respect to Lebesgue measure, and $L^p(E)$ denotes the Lebesgue space of a subset E of $\mathbb{R}^d$ with respect to Lebesgue measure.
- Partial credit will be given for partially correct solutions, even if incomplete.
- The parts of problems are not equally difficult, and will not be weighted equally.
- Good luck!

(1) Let $\mathbb{N}$ be the natural numbers, and $\mathcal{B}$ be the σ-algebra of all subsets of $\mathbb{N}$.

(a) Give an example of a non-negative, non-zero measure μ on $(\mathbb{N}, \mathcal{B})$ such that

$$\mu(\mathbb{N}) = 1. \tag{1}$$

(b) Give an example of a non-negative, non-zero measure μ on $(\mathbb{N}, \mathcal{B})$ such that

$$A \subseteq \mathbb{N}, \quad n \in \mathbb{N} \quad \Rightarrow \quad \mu(A+n) = \mu(A). \tag{2}$$

(Here $A + n := \{a + n \mid a \in A\}$.)

(c) Can there exist a non-negative, non-zero measure μ on $(\mathbb{N}, \mathcal{B})$ with the properties in lines (1) and (2) simultaneously? Justify your answer.

(2) For what values of $p \in (0, \infty)$ is the function $f(x, y) = (x^4 + y^2)^{-p}$ integrable over the punctured disk $D' = \{(x, y) \in \mathbb{R}^2 \mid 0 < x^2 + y^2 \leqslant 1\}$? *Hint: consider the integrals* $I_n = \int_{A_n} f(x, y)dxdy$, *where* $A_n = \{(x, y) \in D' \mid 2^{-n-1} \leqslant x^4 + y^2 \leqslant 2^{-n}\}$; *how are the various* I_n *related?*

(3) Fix $p \in (1, \infty)$. Of the following two statements, one is true, and one is false. Prove the true one, and disprove the false one.

(a) There exists a constant $C > 0$ such that for all simple functions $f : [1, \infty) \to \mathbb{C}$

$$\int_1^\infty |f(x)| dx \leqslant C \Big(\int_1^\infty |f(x)|^p dx \Big)^{1/p}.$$

(b) There exists a constant $C > 0$ such that for all simple functions $f : [1, \infty) \to \mathbb{C}$

$$\int_1^\infty |f(x)| dx \leqslant C \Big(\int_1^\infty |x f(x)|^p dx \Big)^{1/p}.$$

(4) Let $C[-1, 1]$ denote the space of continuous functions $f : [-1, 1] \to \mathbb{R}$, equipped with the supremum norm. Let (f_n) be a sequence of in $C[-1, 1]$ with the following properties:

- each f_n is differentiable (with one-sided derivatives at the end points) and $|f_n'(x)| \leqslant 1$ for all n, and all $x \in [-1, 1]$;
- the sequence (f_n) converges in norm to some $f \in C[-1, 1]$.

(a) Show that f satisfies the inequality $|f(x) - f(y)| \leqslant |x - y|$ for all $x, y \in [-1, 1]$.

(b) Give an example of (f_n) and f with the properties above such that f is not differentiable. You do not have to justify your answer, and will get partial credit for persuasive pictures and explanation.

(c) Prove that a general f as in the statement is differentiable almost everywhere on $[-1, 1]$.

(5) Show that for any $f \in L^1(\mathbb{R})$,

$$\int_{\mathbb{R}} f(x) e^{itx} dx \to 0 \text{ as } t \to \infty.$$

(6) Let (f_n) be a sequence of functions in $L^1(\mathbb{R})$ which converges to zero in L^1-norm. For each of the following statements, either prove it, or give a counterexample.

(a) (f_n) converges to zero almost everywhere.

(b) (f_n) converges to zero in measure.

(c) For any $g \in L^1(\mathbb{R})$, $\|f_n * g\|_1$ tends to zero. Here $f * g$ denotes the convolution defined by

$$(f * g)(x) = \int_{\mathbb{R}} f(x-t)g(t)dt;$$

you may assume without proof that $f * g$ is a well-defined element of $L^1(\mathbb{R})$ when both f and g are in $L^1(\mathbb{R})$.