ANALYSIS QUALIFYING EXAM, SPRING 2018

- Throughout this exam, unless otherwise indicated, m denotes the Lebesgue measure in $\mathbb{R}^d$, integration is with respect to m, and $L^p(E)$ denotes the Lebesgue space (real-valued functions) of a subset E of $\mathbb{R}^d$ with respect to m.
- Partial credit will be given for partially correct solutions, even if incomplete.
- The parts of problems are not equally difficult, and will not be weighted equally.
- Good luck!

Problem 1

Let $\{r_k\}_{k=1}^{\infty}$ be an enumeration of the rational numbers in $(0,1]$, and define

$$f(x) := \sum_{k=1}^{\infty} \frac{1}{k^2|x-r_k|^{1/2}} \qquad (0 \le x \le 1).$$

Prove that f is finite a.e. with respect to Lebesgue measure on $[0,1]$.

Problem 2

Let $f \in L^2([0,1])$, and suppose that

$$\int_0^1 f(x)x^n \, dx = 0 \qquad (n = 0, 1, 2, \dots)$$

Show that $f = 0$ a.e. with respect to Lebesgue measure on $[0,1]$.

Problem 3

Let μ and ν be finite positive measures on a set X such that $\mu \ll \nu$, and let $\frac{d\nu}{d(\mu+\nu)}$ be the Radon-Nikodym derivative of ν with respect to $\mu + \nu$. Show that

$$0 < \frac{d\nu}{d(\mu+\nu)} < 1$$

almost everywhere with respect to μ.

Problem 4

Let μ be a positive finite measure on a set X with $\mu(X) > 0$, and let $f \in L^\infty(\mu)$ with $\|f\|_\infty > 0$.

a. Show that $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$.

b. For $n \in \mathbb{N}$, let $\alpha_n := \int_X |f|^n \, d\mu$. Show that

$$\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = \|f\|_\infty.$$

Problem 5

a. Let $F \subset \mathbb{R}^d$ be measurable. Prove that for all $x \in \mathbb{R}^d$,

$$m(F) = \int_{\mathbb{R}^d} \chi_F(x - t) \, dt.$$

Here χ_F denotes the characteristic function of the set F.

b. Let E and F be measurable subsets of $\mathbb{R}^d$ with $m(E)m(F) > 0$. Prove that there is a translate of F that intersects E in a set of positive measure. Here a *translate* of F is a set of the form $F + t = \{f + t : f \in F\}$ for some $t \in \mathbb{R}^d$.

Problem 6

Let $1 < p < \infty$, and let (f_n) be a sequence of functions in $L^p([0,1])$ that converges almost everywhere to a function $f \in L^p([0,1])$. Suppose in addition that there is a constant M such that $\|f_n\|_p \leq M$ for all n.

Show that for each $g \in L^q([0,1])$, where $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\lim_{n\to\infty} \int_0^1 f_n g = \int_0^1 fg.$$