ANALYSIS QUALIFYING EXAM - JANUARY 2019

Attempt the following six problems. Please note the following:

- Throughout this exam, $L^p(X)$ denotes the L^p-space of a measure space $(X, \mathcal{M}, \mu)$ with the associated norm of a function being $\|f\|_p$. Subsets of $\mathbb{R}^d$ are equipped with the Lebesgue sigma algebra and Lebesgue measure unless otherwise stated.
- Partial credit will be given for partially correct solutions, even if incomplete.
- The parts of problems are not equally difficult, and will not be weighted equally.
- Good luck!

(1) Let $\{ f_n \}$ be a sequence of non-negative real valued measurable functions on a measure space $(X, \mathcal{M}, \mu)$. Suppose that

$$\sum_{n=1}^{\infty} \mu\left(\{ x \in X : f_n(x) > 1 \}\right) < \infty.$$

Show that for μ-almost every $x \in X$, $\limsup_{n\to\infty} f_n(x) \leq 1$.

(2) Show that if $f \in L^p(\mathbb{R})$ for some $p \in [1, \infty)$ then

$$\lim_{t\to\infty} t^p \cdot m\left(\{ x : |f(x)| > t \}\right) = 0.$$

(3) Let $f \in L^1(\mathbb{R})$. Prove that

$$\lim_{t \to 0} \int_{\mathbb{R}} |f(x+t) - f(x)| \, dx \quad = 0.$$

(4) Define $f_k : \mathbb{R} \to \mathbb{R}$ by $f_k(x) = x^k e^{-x^2/2}$, for $k = 0, 1, \dots$.

(a) Prove that $\|f_k\|_2^2 \leq k!$ for all $k \in \mathbb{N}$.

You may assume without proof that $\|f_0\|_2^2 \leq 1$.

It may help to observe that $\|f_k\|_2^2 = \int_0^\infty y^{k-1/2} e^{-y} dy$ *for* $k \geq 1$.

(b) Show that for any $\lambda \in \mathbb{R}$, the series

$$\sum_{k=0}^{\infty} \frac{i^k \lambda^k f_k}{k!}$$

converges in $L^2(\mathbb{R})$ to the function $g_\lambda(x) = e^{i\lambda x - x^2/2}$.

You may assume without proof that $n! \geq \sqrt{2\pi} n^{n+1/2} e^{-n}$*; this is a consequence of Stirling's approximation.*

(5) Let $f \in L^1(\mathbb{R})$, and let $g \in L^p(\mathbb{R})$ for some $p \in [1, \infty)$. Define their convolution by

$$(f * g)(x) := \int_{\mathbb{R}} f(y) g(x-y) dy.$$

Assuming that the function $y \mapsto f(y)g(x-y)$ is measurable for fixed $x \in \mathbb{R}$, show that $(f * g)(x)$ is well-defined for almost all $x \in \mathbb{R}$, that the resulting function is in $L^p(\mathbb{R})$, and that $\|f * g\|_p \leq \|f\|_1 \|g\|_p$.

(6) Let μ and ν be finite, positive measures on a measurable space $(X, \mathcal{M})$. Prove that the following are equivalent:

(a) ν is absolutely continuous with respect to μ.

(b) for every $\varepsilon > 0$ there exists $\delta > 0$ with the following property: if $A \in \mathcal{M}$ and $\mu(A) < \delta$ then $\nu(A) < \varepsilon$.