ANALYSIS QUALIFYING EXAM - JANUARY 2020

Attempt the following six problems. Please note the following:

- Throughout this exam, $L^p(X)$ denotes the L^p-space of a measure space $(X, \mathcal{M}, \mu)$ with the associated norm of a function being $\|f\|_p$. Subsets of $\mathbb{R}^d$ are equipped with the Lebesgue sigma algebra and Lebesgue measure m unless otherwise stated.
- Partial credit will be given for partially correct solutions, even if incomplete.
- The parts of problems are not equally difficult, and will not be weighted equally.
- Good luck!

(1) (a) Give an example of a sequence (f_n) of continuous functions $f_n : [0,1] \to \mathbb{R}$ and a continuous function $f : [0,1] \to \mathbb{R}$ such that $f_n(x) \to f(x)$ for all $x \in [0,1]$, but such that

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx \neq \int_0^1 f(x)dx.$$

Justify your answer for full credit.

(b) Assume that (f_n) and f satisfy the hypotheses of (a), and that there exists $M \in (0,\infty)$ so that $|f_n(x)| \leq M$ for all $x \in [0,1]$. Show that

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx.$$

(2) Show that for each $\epsilon > 0$ there exists a subset A of $[0,1] \times [0,1]$ with the following properties:
(a) A is closed;
(b) $m(A) > 1 - \epsilon$;
(c) for all $(x, y) \in A$, both x and y are irrational.

(3) Let $f : \mathbb{R}^d \to \mathbb{R}$ be a measurable function, and let $G := \{(x, y) \in \mathbb{R}^d \times \mathbb{R} \mid y = f(x)\}$ be its graph.
  (a) Show that G is measurable (for the Lebesgue σ-algebra on $\mathbb{R}^{d+1}$).
  (b) Show that $m(G) = 0$.

(4) A measure space $(X, \mathcal{M}, \mu)$ is said to be *σ-finite* if there is a countable collection (E_n) of measurable sets such that $\mu(E_n) < \infty$ for all n, and so that $X = \cup_{n=1}^{\infty} E_n$.
  (a) Give an example of a measure space that is not σ-finite. Justify your answer for full credit.
  (b) Show that $(X, \mathcal{M}, \mu)$ is σ-finite if and only if there exists an integrable function f on X such that $f(x) > 0$ for all $x \in X$.

(5) Let $f \in L^1(\mathbb{R}^2)$. For each $(\xi, \eta) \in \mathbb{R}^2$, let

$$a(\xi, \eta) := \int_{\mathbb{R}^2} f(x, y) e^{i(x\xi + y\eta)} dx dy.$$

  (a) Show that the function

$$\mathbb{R}^2 \to \mathbb{C}, \quad (\xi, \eta) \mapsto a(\xi, \eta)$$

  is well-defined and continuous.
  (b) Show that for all $\epsilon > 0$ there is a compact set $K \subseteq \mathbb{R}^2$ such that if $(\xi, \eta) \notin K$, then $|a(\xi, \eta)| < \epsilon$.

(6) (a) Show that neither of $L^4(\mathbb{R})$ nor $L^2(\mathbb{R})$ contains the other.
  (b) Which of $L^2([0, 1])$ and $L^4([0, 1])$ contains the other? Justify your answer for full credit.
  (c) Let $\mathbb{N}$ be equipped with counting measure. Which of $L^2(\mathbb{N})$ and $L^4(\mathbb{N})$ contains the other? Justify your answer for full credit.