

ANALYSIS QUALIFYING EXAM - JANUARY 2021

Attempt the following six problems. Please note the following:

- Throughout this exam, unless otherwise indicated, m denotes the Lebesgue measure in \mathbb{R}^d , integration is with respect to m , and $L_p(E)$ denotes the Lebesgue space of a subset E of \mathbb{R}^d with respect to m .
- Partial credit will be given for partially correct solutions, even if incomplete.
- The parts of problems are not equally difficult, and will not be weighted equally.
- Good luck!

(1) Let μ be a finite measure on X and let $f : X \rightarrow (0, \infty)$ be a positive, measurable function. Show that, for any $\varepsilon > 0$, there exist $\delta > 0$ and a measurable subset E of X such that $f \geq \delta$ on E and $\mu(X \setminus E) < \varepsilon$.

(2) Give an example of a function $f \in L_1(\mathbb{R})$ which satisfies the following property: for every nontrivial interval I , $f \cdot \chi_I \notin L_\infty(\mathbb{R})$. (Here χ_I is the characteristic function of the interval I .) Be sure to explain why your example works.

(3) Let \mathcal{P}_e denote the family of all even polynomials. Thus a polynomial $p \in \mathcal{P}_e$ if and only if $p(x) = p(-x)$ for all x . Prove that the closure of \mathcal{P}_e in $L_1([-1, 1])$ is

$$A := \{f \in L_1([-1, 1]) \mid f(x) = f(-x) \text{ a.e.}\}.$$

You may use without proof that the continuous functions on $[-1, 1]$ are dense in $L_1([-1, 1])$.

- (4) In this problem, let \mathcal{B} denote the Borel σ -algebra on \mathbb{R} .
- (a) Provide an example of a signed Borel measure $\nu : \mathcal{B} \rightarrow \mathbb{R}$ whose range is not connected.
 - (b) Show that if λ is a finite, signed Borel measure for which $m(A) = 0$ implies $\lambda(A) = 0$, then the range of λ is compact and connected.

- (5) Suppose $k : \mathbb{R} \rightarrow \mathbb{R}$ is a bounded, continuous function with the additional property that

$$\sup_{-\infty < a < b < \infty} \left| \int_a^b k(x) dx \right| < \infty.$$

(Sine and cosine are examples of such functions.) For $f \in L_1(\mathbb{R})$, define the function $Kf : \mathbb{R} \rightarrow \mathbb{R}$ by:

$$Kf(x) = \int_{\mathbb{R}} k(xy)f(y) dy, \quad x \in \mathbb{R}.$$

Prove that for every $f \in L_1(\mathbb{R})$,

- (a) Kf is continuous on \mathbb{R} , and
- (b) $\lim_{|x| \rightarrow \infty} Kf(x) = 0$.

Suggestion: For part (b), first consider the case that f is the characteristic function of a finite interval.

- (6) Let μ be a finite measure on X and suppose that the sequence $\{f_n\}_{n=1}^{\infty} \subset L_2(\mu)$ satisfies $\sup_n \|f_n\|_2 = M < \infty$ and $f_n(x) \rightarrow f(x)$ a.e. Prove that $f_n \rightarrow f$ in $L_1(\mu)$.