

ANALYSIS QUALIFYING EXAM, SPRING 2022

- Throughout this exam (X, μ) denotes an arbitrary measure space, and $L^p(\mu)$ the corresponding Lebesgue space. Also, m denotes the Lebesgue measure in \mathbb{R}^d , and $L^p(E)$ denotes the Lebesgue space (real-valued functions) of a subset E of \mathbb{R}^d with respect to m .
- Partial credit will be given for partially correct solutions, even if incomplete.
- The parts of problems are not equally difficult, and will not be weighted equally.
- Good luck!

Problem 1

Let $E \subset \mathbb{R}^d$ be a Lebesgue measurable set with $m(E) > 0$. Show that there exists $x \in E$ such that for all $\delta > 0$, we have $m(E \cap B(x, \delta)) > 0$, where $B(x, \delta)$ denotes the open ball centered at x of radius δ .

Problem 2

For $f \in L^1(\mathbb{R})$, define a function \hat{f} by

$$\hat{f}(t) := \int_{-\infty}^{\infty} f(x) e^{-ixt} dm(x) \quad (t \in \mathbb{R}).$$

- Show that \hat{f} is continuous and bounded on \mathbb{R} .
- Show that $\hat{f}(t) \rightarrow 0$ as $|t| \rightarrow \infty$.

Problem 3

Let $f : X \rightarrow [0, \infty]$ be measurable. Suppose that $c := \int_X f d\mu$ satisfies $0 < c < \infty$, and let $0 < \alpha \leq 1$. Prove that

$$\lim_{n \rightarrow \infty} \int_X n \log \left(1 + \left(\frac{f}{n} \right)^\alpha \right) d\mu = \begin{cases} \infty & \text{if } 0 < \alpha < 1 \\ c & \text{if } \alpha = 1. \end{cases}$$

Problem 4

Let $E \subset \mathbb{R}$ be Lebesgue measurable. Define

$$F := \{x \in \mathbb{R} : m(I_x \cap E) > 0\},$$

where $I_x := (x - 1, x + 1)$. Show that F is Lebesgue measurable.

Problem 5

Let (f_n) be a sequence of measurable real-valued functions on \mathbb{R} , and for $\epsilon > 0$, let

$$E_{n,\epsilon} := \{x \in \mathbb{R} : |f_n(x)| > \epsilon\}.$$

a. Show that if $\sum_n m(E_{n,\epsilon}) < \infty$ for each $\epsilon > 0$, then $f_n \rightarrow 0$ almost everywhere on \mathbb{R} .

b. Show that if $(f_n) \subset L^p(\mathbb{R})$ for some $p \in [1, \infty)$ and if $f_n \rightarrow 0$ in $L^p(\mathbb{R})$, then $m(E_{n,\epsilon}) \rightarrow 0$ as $n \rightarrow \infty$, for each $\epsilon > 0$.

Problem 6

Let $f, g : X \rightarrow (0, \infty)$ be measurable functions with $\int f d\mu = \int g d\mu = 1$. Show that if $f \log g \in L^1(\mu)$, then

$$\int f \log f d\mu \geq \int f \log g d\mu.$$