ANALYSIS QUALIFYING EXAM, SPRING 2023

- Throughout this exam (X, μ) denotes an arbitrary measure space, and $L^p(\mu)$ the corresponding Lebesgue space. Also, m denotes the Lebesgue measure in $\mathbb{R}^d$, and $L^p(E)$ denotes the Lebesgue space (real-valued functions) of a subset E of $\mathbb{R}^d$ with respect to m.
- Partial credit will be given for partially correct solutions, even if incomplete.
- The parts of problems are not equally difficult, and will not be weighted equally.
- Good luck!

Problem 1

a. Let $f : X \to \mathbb{R}$ be measurable, and let $p \in (0, \infty)$. For $n \in \mathbb{N}$, define

$$E_n := \{n - 1 \leq |f| < n\}.$$

Show that if $\mu(E_1) < \infty$, then $f \in L^p(\mu)$ if and only if $\sum_{n=1}^{\infty} n^p \mu(E_n) < \infty$.

b. Deduce that $\log \in L^p((0, 1))$ for all $p \in (0, \infty)$.

Problem 2

a. Let $B \subset \mathbb{R}^n$ be Borel. Prove that for all $x \in \mathbb{R}^n$,

$$m(B) = \int_{\mathbb{R}^n} \chi_B(x - t)\, dt.$$

b. Let A and B be Borel subsets of $\mathbb{R}^n$ with $m(A)m(B) > 0$. Prove that there is a translate of B that intersects A in a set of positive Lebesgue measure. Here a *translate* of B is a set of the form $B + t = \{b + t : b \in B\}$ for some $t \in \mathbb{R}^n$.

Problem 3 Consider the function $f(x, y) := 2e^{-2xy} - e^{-xy}$ defined on $[0, \infty) \times [0, 1]$.

a. Show that

$$\int_0^1 \int_0^{\infty} f(x, y)\, dx\, dy = 0.$$

b. Show that
$$\int_0^\infty \int_0^1 f(x,y)\,dy\,dx = \log 2.$$
c. What can we deduce about f? Explain.

Problem 4 Let $f \in L^1([0,1])$, $f > 0$. Which of the numbers
$$\int_0^1 f \log f\, dm$$
or
$$\left(\int_0^1 f\, dm\right)\left(\int_0^1 \log f\, dm\right)$$
is the larger?
(*Hint:* Use Jensen's inequality.)

Problem 5 Suppose that $\mu(X) = 1$, and let $(A_n)_{n\in\mathbb{N}}$ be a sequence of measurable subsets of X. Recall that $(A_n)_{n\in\mathbb{N}}$ is *independent* if
$$\mu\left(\bigcap_{j=1}^n A_{i_j}\right) = \prod_{j=1}^n \mu(A_{i_j})$$
for all $i_1, \ldots, i_n \in \mathbb{N}$.

a. Show that if $(A_n)_{n\in\mathbb{N}}$ is independent, then so is $(A_n^c)_{n\in\mathbb{N}}$, where A_n^c is the complement of A_n in X.

b. Suppose that $(A_n)_{n\in\mathbb{N}}$ is independent and that $\sum_{n=1}^\infty \mu(A_n) = \infty$. Show that $\mu(\limsup_{n\to\infty} A_n) = 1$, where
$$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^\infty \bigcup_{k=n}^\infty A_k.$$
(*Hint:* Let $A := \limsup_{n\to\infty} A_n$. Show that $\mu(A^c) = 0$. You might want to use part **a** and the inequality $1 - x \le e^{-x}$ valid for $x \ge 0$.)

Problem 6 Let $X = [0,1]$ and let $\mathcal{B} = \mathcal{B}([0,1])$ be the σ-algebra of Borel subsets of $[0,1]$. Let μ be the counting measure on $\mathcal{B}$, i.e., $\mu(E)$ equals the cardinality of E for $E \in \mathcal{B}$.

a. Show that m is absolutely continuous with respect to μ.

b. Show that there is no $f \in L^1(\mu)$ such that $dm = f\,d\mu$.

c. Does this contradict the Radon–Nikodym theorem? Explain.