MOCK ANALYSIS QUALIFYING EXAM 1

Attempt the following six problems. Please note the following:

- Throughout the exam, unless indicated otherwise, integration is with respect to Lebesgue measure.
- We denote the Lebesgue measure of a set A by $m(A)$.

(1) Let A, B, C, D, E be measurable subsets of $[0,1]$. Suppose that almost every $x \in [0,1]$ belongs to at least 4 of these subsets. Prove that at least one of the sets has measure of at least $4/5$.

(2) Suppose $f \in L_1(\mathbb{R})$ and there is $0 < \epsilon$ so that $|\int_A f(x)\mathrm{d}x| \leq \big(m(A)\big)^{1+\epsilon}$ for all measurable sets $A \subset \mathbb{R}$. Prove that $f = 0$ a.e.

(3) (a) Find a sequence of functions f_n in $L_1([0,1])$ converging a.e. to a limit g so that $\|f_n\|_1 = 2$ for all n and $\|g\|_1 = 1$.

(b) Prove that if f_n and g are in L_1 and satisfy the conditions of part (a), then

$$\lim_{n\to\infty} \int_0^1 |f_n(x) - g(x)|\mathrm{d}x = 1.$$

(4) Suppose $f : [0,1] \to \mathbb{C}$ is measurable and satisfies $0 < \int_0^1 |f(x)|\mathrm{d}x < \infty$. Suppose $0 < \epsilon < 1$. Show that

$$\lim_{n\to\infty} n \int_0^1 \log\left(1 + \left(\frac{|f|}{n}\right)^{\epsilon}\right) \mathrm{d}x = \infty$$

(5) Consider the "hat" function $\phi \in C(\mathbb{R})$, defined by

$$\phi(x) = \begin{cases} 1 - |x| & x \in [-1, 1] \\ 0 & \text{otherwise} \end{cases}$$

(a) Show that if $g \in C^1(\mathbb{R})$ (i.e. g is continuously differentiable), then $g * \phi$ is continuously differentiable.

(b) Find a function $F \in L_\infty(\mathbb{R})$ so that if $g \in C^1(\mathbb{R})$ then $\frac{d}{dx}(g * \phi) = g * F$.

(c) Show that if $f \in L_1(\mathbb{R})$ then $f * \phi \in C^1(\mathbb{R})$.

(6) Let C denote the Cantor set: i.e., the set of all reals $0 \leq x \leq 1$ that can be expanded in base 3 using only digits 0 and 2. In other words, $x \in C$ if and only if $x = \sum_{j=1}^{\infty} c_j 3^{-j}$ with $c_j \in \{0, 2\}$ for each j.

The Cantor function $F : [0, 1] \to \mathbb{R}$ is a continuous function defined as follows: for $x = \sum_{j=1}^{\infty} c_j 3^{-j} \in C$, $F(x) = \sum_{j=1}^{\infty} c_j 2^{-j-1}$, while on each interval in $[0, 1] \setminus C$, F is constant. Evaluate

$$\int_0^1 F(x) \mathrm{d}x.$$