MOCK ANALYSIS QUALIFYING EXAM 3

Attempt the following six problems. Please note the following:

- Throughout the exam, unless indicated otherwise, integration is with respect to Lebesgue measure.
- We denote the Lebesgue measure of a set A by $m(A)$.

(1) Suppose (f_n) is a sequence of measurable, complex valued functions on a measure space Ω. Show that the the function $h : \Omega \to \mathbb{N} \cup \{\infty\}$ defined by

$$h(x) = \#\{n \mid f_n(x) = 0\}$$

is measurable.

(2) Suppose $f \in L_1([0, 1)$ and that f_n is a sequence of functions converging to f in L_1, but satisfying that each $f_n \in L_2([0, 1])$ with $\|f_n\|_2 \leq 1$.

(a) Show that $f \in L_2([0, 1])$ and $\|f\|_2 \leq 1$.

(b) Give an example satisfying the above hypotheses, where f_n does not converge to f in the $L_2([0, 1])$ topology.

(3) (a) Suppose $f \geq 0$ on $[0, 1]$. Show that for all $0 < p < \infty$,

$$\left(\int_0^1 f(x)^p \mathrm{d}x\right)^{1/p} \geq e^{\int_0^1 \log f(x) \mathrm{d}x}.$$

(Where we consider $e^{-\infty} = 0$.)

(b) Suppose f and g are measurable and non-negative throughout $[0, 1]$. For which pairs of numbers $p, q > 0$ does $f(x)g(x) \geq 1$ imply

$$\left(\int_0^1 f(x)^p \mathrm{d}x\right)^{1/p} \left(\int_0^1 g(x)^q \mathrm{d}x\right)^{1/q} \geq 1?$$

(4) Let K be a measurable subset of $\mathbb{R}$,and for each $n \in \mathbb{N}$ let

$$U_n = \{x \in \mathbb{R} \mid |x - y| \leq \frac{1}{n} \text{ for some } y \in K\}.$$

(a) Show that if K is closed and bounded then

$$\lim_{n \to \infty} m(U_n) = m(K) \tag{0.1}$$

(b) Show that the equality in line (0.1) need not hold if K is closed but not bounded.

(c) Show that the equality in line (0.1) need not hold if K is bounded but not closed.

(5) Show that

$$\lim_{n \to \infty} \int_0^\infty \frac{1 - e^{-nx}}{1 + x^2} \cos(n\pi x) \mathrm{d}x = 0.$$

(6) Let δ_x denote the point mass at x, i.e.,

$$\delta_x(E) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

For this problem, let m denote Lebesgue measure on $\mathbb{R}$, and let μ be the Borel measure $\mu = m + \delta_0 + \delta_1$. Let λ be a signed Borel measure on $\mathbb{R}$ such that for any continuously differentiable, compactly supported function $f : \mathbb{R} \to \mathbb{R}$ we have

$$\int f(x) \mathrm{d}\lambda(x) = \int_0^1 f'(x) x^2 \mathrm{d}m(x)$$

Prove that λ is absolutely continuous with respect to μ and find the Radon-Nikodym derivative $\frac{\mathrm{d}\lambda}{\mathrm{d}\mu}$.