MOCK ANALYSIS QUALIFYING EXAM 4

Attempt the following six problems. Please note the following:

- Throughout the exam, unless indicated otherwise, integration is with respect to Lebesgue measure.
- We denote the Lebesgue measure of a set A by $m(A)$.

(1) Let Ω be a subset of $\mathbb{R}^d$ with finite Lebesgue measure and $f : \Omega \to \mathbb{R}$ a non-negative, integrable function. Assume that

$$\int_\Omega f(x)\mathrm{d}x = \int_\Omega \big(f(x)\big)^n \mathrm{d}x$$

for all $n \in \mathbb{N}$. Prove that there exists a measurable set $E \subset \Omega$ such that $f = \chi_E$ a.e.

(2) Suppose (f_n) is a sequence of non-negative functions in $L_2([0,1])$, satisfying $\|f_n\|_1 = 1$ for all $n \in \mathbb{N}$. Assume, further, that

$$\big|\|f_n\|_2 - 1\big| \leq 2^{-n}.$$

Show that $f_n \to 1$ almost everywhere.

(3) (a) Give an example of a measurable function $f : [0,1] \to \mathbb{R}$ which is nowhere continuous, but equal almost everywhere to a continuous function.

(b) Let E be a closed subset of $[0,1]$ with positive measure and dense complement (such sets exist - you do not have to justify this). Show that if $F \subseteq [0,1]$ has measure one, then the restriction of χ_E to F is not continuous.

(4) Let $\alpha \in \mathbb{R}$ and $\vec{x} \in \ell_2(\mathbb{N})$. Find a sequence $\vec{x}_j = (x_{j,k})_{j\in\mathbb{N}}$ of elements of $\ell_2(\mathbb{N})$ so that $\vec{x}_j \to \vec{x}$ in $\ell_2(\mathbb{N})$ and each $\vec{x}_j$ has only finitely many nonzero entries and so that $\sum_k x_{j,k} = \alpha$.

(5) Let $f \in L_1([0,1])$ and define, for $k \in \mathbb{N}$, the step function f_k where

$$f_k(x) = k \int_{j/k}^{(j+1)/k} f(t)\mathrm{d}t \quad \text{for } x \in \left[\frac{j}{k}, \frac{j+1}{k}\right].$$

(a) Show that $f_k \to f$ in $L_1([0,1])$.

(b) Consider $f(x) = \frac{1}{x(\ln x)^2}$. Show that $(f_k)_{k\in\mathbb{N}}$ does not satisfy the conditions of the Dominated Convergence Theorem.

(6) For what values of $p \in (0,\infty)$ is $f(x,y) = (x^2+y^4)^{-p}$ integrable over the punctured plane $\{(x,y) \mid x^2+y^2 \geq 1\}$?
Hint: consider first the integrals $I_n = \int_{A_n}(x^2+y^4)^{-p}\mathrm{d}A$ *over the sets* $A_n = \{(x,y) \mid 2^n \leq (x^2+y^4) < 2^{n+1}$. *How are they related?*