Attempt the following six problems. Please note the following:

- Throughout the exam, unless indicated otherwise, integration is with respect to Lebesgue measure.
- We denote the Lebesgue measure of a set \( A \) by \( m(A) \).

(1) Let \( \Omega \) be a subset of \( \mathbb{R}^d \) with finite Lebesgue measure and \( f : \Omega \to \mathbb{R} \) a non-negative, integrable function. Assume that

\[
\int_{\Omega} f(x)\,dx = \int_{\Omega} (f(x))^n\,dx
\]

for all \( n \in \mathbb{N} \). Prove that there exists a measurable set \( E \subset \Omega \) such that \( f = \chi_E \) a.e.

(2) Suppose \((f_n)\) is a sequence of non-negative functions in \( L_2([0, 1]) \), satisfying \( \|f_n\|_1 = 1 \) for all \( n \in \mathbb{N} \). Assume, further, that

\[
\|f_n\|_2 - 1 \leq 2^{-n}.
\]

Show that \( f_n \to 1 \) almost everywhere.

(3) (a) Give an example of a measurable function \( f : [0, 1] \to \mathbb{R} \) which is nowhere continuous, but equal almost everywhere to a continuous function.

(b) Let \( E \) be a closed subset of \([0, 1]\) with positive measure and dense complement (such sets exist - you do not have to justify this). Show that if \( F \subseteq [0, 1] \) has measure one, then the restriction of \( \chi_E \) to \( F \) is not continuous.

(4) Let \( \alpha \in \mathbb{R} \) and \( \bar{x} \in \ell_2(\mathbb{N}) \). Find a sequence \( \bar{x}_j = (x_{j,k})_{j \in \mathbb{N}} \) of elements of \( \ell_2(\mathbb{N}) \) so that \( \bar{x}_j \to \bar{x} \) in \( \ell_2(\mathbb{N}) \) and each \( \bar{x}_j \) has only finitely many nonzero entries and so that \( \sum_k x_{j,k} = \alpha \).
(5) Let \( f \in L_1([0, 1]) \) and define, for \( k \in \mathbb{N} \), the step function \( f_k \) where
\[
f_k(x) = k \int_{j/k}^{(j+1)/k} f(t) \, dt \quad \text{for} \quad x \in \left[ \frac{j}{k}, \frac{j + 1}{k} \right].
\]
(a) Show that \( f_k \to f \) in \( L_1([0, 1]) \).

(b) Consider \( f(x) = \frac{1}{x \ln x} \). Show that \( (f_k)_{k \in \mathbb{N}} \) does not satisfy the conditions of the Dominated Convergence Theorem.

(6) For what values of \( p \in (0, \infty) \) is \( f(x, y) = (x^2 + y^4)^{-p} \) integrable over the punctured plane \( \{(x, y) \mid x^2 + y^2 \geq 1\} \)?
\[\text{Hint: consider first the integrals } I_n = \int_{A_n} (x^2 + y^4)^{-p} \, dA \text{ over the sets } A_n = \{(x, y) \mid 2^n \leq (x^2 + y^4) < 2^{n+1}\}. \text{ How are they related?}\]