Attempt the following six problems. Please note the following:

- Throughout the exam, unless indicated otherwise, integration is with respect to Lebesgue measure.
- We denote the Lebesgue measure of a set $A$ by $m(A)$.

(1) Let $m$ denote Lebesgue measure on $\mathbb{R}^3$, and let

$$E := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + 2y^2 + 4z^2 = 1\}.$$ 

Define a subset $A$ of $E$ to be $E$-measurable if

$$A_0 := \{ta \in \mathbb{R}^3 \mid t \in [0, 1], a \in A\}$$

is measurable in $\mathbb{R}^3$, and for an $E$-measurable set $A \subseteq E$ define

$$m_E(A) := m(A_0).$$

(a) Show that the collection of $E$-measurable sets is a $\sigma$-algebra, that $m_E$ is a measure on $E$, and that every continuous function $f : E \to \mathbb{R}$ is measurable for this $\sigma$-algebra.

(b) Compute $m_E(E)$ (you may assume the measure of the unit ball in $\mathbb{R}^3$ is $\frac{4}{3}\pi$).

(2) (a) Say $f : [0, 1] \to \mathbb{R}$ is continuous, and differentiable on $(0, 1)$ with bounded derivative. Show that $f$ has bounded variation.

(b) For each $a > 0$, define

$$f_a : [0, 1] \to \mathbb{R}, \quad f_a(x) = \begin{cases} x^a \sin(x^{-2}) & x > 0, \\ 0 & x = 0. \end{cases}$$

For which values of $a$ does $f_a$ have bounded variation? Justify your answer.
(3) Let \( f : \mathbb{R} \to (0, \infty) \) be defined by \( f(x) = x^{-1/2} \chi_{(0,1)}(x) \). Let \((q_n)_{n=1}^{\infty}\) be an enumeration of the rationals, and define \( F : \mathbb{R} \to [0, \infty] \) by
\[
F(x) = \sum_{n=1}^{\infty} 2^{-n} f(x - q_n).
\]
Show that the series defining \( F \) converges to a finite number almost everywhere, but that \( F \) is unbounded on any (non-empty) interval.

(4) Let \( p > q \) be fixed numbers in \([1, \infty]\). Give a proof or a counterexample to each of the statements below.
(a) If \( f \in L^p(\mathbb{R}) \) has finite measure support, then \( f \in L^q(\mathbb{R}) \).
(b) If \( f \in L^q(\mathbb{R}) \) has finite measure support, then \( f \in L^p(\mathbb{R}) \).
(c) If \( f \in L^p(\mathbb{R}) \) is bounded, then \( f \in L^q(\mathbb{R}) \).
(d) If \( f \in L^q(\mathbb{R}) \) is bounded, then \( f \in L^p(\mathbb{R}) \).

(5) Let \( X \) be a measure space. Prove that the (integrable) simple functions are dense in \( L^\infty(X) \) if and only if \( X \) has finite measure.

(6) For each \( n \), let \( p_n \in [1, \infty] \). Set
\[
E := \left\{ (v_n)_{n=1}^{\infty} \mid v_n \in L^{p_n}(\mathbb{R}) \text{ and } \sum_{n=1}^{\infty} \|v_n\|_{p_n} < \infty \right\}
\]
with the norm
\[
\|(v_n)\| := \sum_{n=1}^{\infty} \|v_n\|_{p_n}
\]
(you may assume without proof that \( E \) is a vector space and this is a norm).
(a) Show that \( E \) is complete.
(b) Let \( v : \mathbb{R} \to \mathbb{R} \) be measurable and non-zero on a set of positive measure. Is it possible that the constant sequence \((v, v, v, ..., v)\) is in \( E \) (for some sequence \((p_n)\))? Give an example of such an \( E \) and \( v \), or a proof that no such \( v \) can exist.