APPLIED MATH QUALIFYING EXAM – SEPTEMBER 2016

Solve all six problems. You have 4 hours. Good luck!

1. Consider a system with n species where the first species is prey for the second species, the second is prey for the third, etc. The population dynamics of such a predator-prey system can be modeled using the following system of differential equations:

$$
\begin{aligned}
\dot{x}_1 &= x_1(r_1 - a_{11}x_1 - a_{12}x_2) \\
\dot{x}_2 &= x_2(-r_2 + a_{21}x_1 - a_{22}x_2 - a_{23}x_3) \\
&\vdots \\
\dot{x}_j &= x_j(-r_j + a_{j,j-1}x_{j-1} - a_{jj}x_j - a_{j,j+1}x_{j+1}) \\
&\vdots \\
\dot{x}_n &= x_n(-r_n + a_{n,n-1}x_{nn-1} - a_{nn}x_n),
\end{aligned}
$$

where x_i denotes the population density of the i-th species, r_i represent the natural growth/death rates of the species, a_{ii} capture the negative effects of intra-species competition, and a_{ij}, $i \neq j$, represent the effect of the predator-prey interactions. All coefficients are positive and the dynamics are restricted to the positive orthant.

Prove that if the above system has an equilibrium point (in the positive orthant) then this point is globally stable.

Hint: Look for a Liapunov function in the form $\sum \alpha_i(x_i - \beta_i \log x_i)$.

2. It is known that if S is a section for a *planar* system of differential equations (i.e. S is a curve transverse to the corresponding vector field) and $p_0, p_1, p_2, \ldots$ is the sequence of intersection points of S with a trajectory of the system ordered monotonically along the trajectory, then $p_0, p_1, \ldots$ is ordered monotonically along S.

   (a) Use the above fact to show that if a planar flow has a closed orbit γ bounding an open region $U \subset \mathbb{R}^2$, then U contains either an equilibrium point or a limit cycle.

   (b) Give an example of a system in $\mathbb{R}^3$ possessing a single closed orbit and having no equilibrium points.

3. Consider the planar system given in polar coordinates by

$$
\begin{aligned}
\dot{r} &= \alpha r - r^3, \\
\dot{\theta} &= \sin\theta + \beta.
\end{aligned}
$$

   (a) Find the conditions on parameters α and β under which the system has: one equilibrium point and no periodic orbits, two equilibrium points and no periodic orbits, no equilibrium points and one periodic orbit. Sketch the phase portrait for each of the cases.

(b) Show that for a fixed $|\beta| > 1$ the system undergoes a Hopf bifurcation at $\alpha = 0$.

(c) Describe bifurcations that occur for fixed $\alpha > 0$ when β is varied. Sketch the corresponding bifurcation diagram.

4. Recall that a discrete dynamical system $x \mapsto g(x), x \in \mathbb{R}^n$, is called chaotic it it possesses a compact invariant set $\Lambda \subset \mathbb{R}^n$ such the $g(x)$ is topologically transitive on Λ and has sensitive dependence on initial conditions on Λ.

   Consider a discrete planar dynamical system $x \mapsto g(x)$, where $g : \mathbb{R}^2 \to \mathbb{R}^2$ is given in polar coordinates by

$$g\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} r \\ 2\pi Q(r) + \theta \end{pmatrix}, \qquad r \in \mathbb{R}^+, \theta \in S^1$$

   where Q is a continuously differentiable function with $\dfrac{dQ}{dr} \neq 0$ for all $r \in \mathbb{R}^+$. Show that this system is not chaotic by proceeding as follows.

   (a) Show that each circle centered around the origin is an invariant set for the above system. Also, show that the system does not have sensitive dependence on initial conditions when restricted to an invariant circle.

   (b) Show that the above system does exhibit sensitive dependence on initial conditions in any compact invariant set with non-empty interior. You may find the following fact useful: if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ then numbers $\{n\alpha\}, n \in \mathbb{N}$, form a dense set in $[0, 1]$, where $\{n\alpha\}$ denotes the fractional part of $n\alpha$.

   (c) Show that the above system is not topologically transitive on any invariant set with non-empty interior.

5. (a) Prove that if A is an $m \times n$ matrix and B is an $n \times m$ matrix with $n < m$, then AB is singular.

   (b) Let A be an $n \times n$ matrix and b be an n-vector. Show that if one of the following two systems has a solution then the other does not:

      i. $Ax = b$

      ii. $A^T y = 0$, $b^T y = 1$

6. Solve the following initial boundary value problem for the heat equation:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, \quad x > 0, t > 0 \\ \frac{\partial u}{\partial x} &= -\sin t, \quad x = 0, t > 0 \\ u &= 0, \quad x > 0, t = 0 \end{aligned}$$

   Show that there is $T > 0$ such that $\sup_{t \geq T} u(0, t) \neq \sup_{t \geq T} (-\sin t)$.