SAMPLE APPLIED MATH QUALIFYING EXAM 1

1. Consider Lotka-Volterra equations for n species:

$$\dot{x}_i = x_i \left(r_i + \sum_{j=1}^{n} a_{ij} x_j \right), \quad i = 1, \ldots, n$$

where x_i denotes the density of the i-th species, r_i is its intrinsic growth/decay rate, and the matrix $A = (a_{ij})$ is the interaction matrix.

   (a) Show that the positive orthant, $\mathbb{R}^n_+$ is an invariant set.

   (b) Prove that if there is an ω limit point in the interior of $\mathbb{R}^n_+$ then the system of equations

$$r_i + \sum_{j=1}^{n} a_{ij} x_j = 0, \quad i = 1, \ldots, n$$

   has a solution with all positive entries.
   Hint: Use the fact that if $\dot{x} = f(x)$ is a vector field on an (invariant) open set $O \subset \mathbb{R}^n$ with flow $\phi(t, x)$, and $V : O \to \mathbb{R}$ is a C^1 function such that $\frac{d(V \circ \phi(t,x))}{dt} \geq 0$ then set of ω limit points in O is contained in the set where $\nabla V \cdot f(x) = 0$

2. Consider the following system on $\mathbb{R}^2$:

$$\dot{x} = \mu x - y - x(x^2 + y^2)$$
$$\dot{y} = x + \mu y - y(x^2 + y^2)$$

where $\mu > 0$.

   (a) Show that in polar coordinates the periodic orbit is given by $\phi_t(\sqrt{\mu}, \theta_0)$, where ϕ_t denote the flow of the system.

   (b) Verify that

$$\Sigma = \left\{ (r, \theta) \in \mathbb{R} \times S^1 | r > 0, \theta = \theta_0 \right\}$$

   is a cross section for the given flow and find the Poincare map $P : \Sigma \to \Sigma$. Show that it has an asymptotically stable fixed point at $r = \sqrt{\mu}$.

3. In a model of a hypothetical chemical oscillator, the dimensionless concentrations $x, y \geq 0$ evolve over time according to

$$\begin{aligned} \dot{x} &= 1 - (b+1)x + ax^2 y, \\ \dot{y} &= bx - ax^2 y, \end{aligned}$$

where $a, b > 0$ are parameters.

(a) Find all the fixed points, and perform their linear stability analysis.

(b) Show that a Hopf bifurcation occurs at some parameter value $b = b_c$, where b_c is to be determined.

(c) Does the limit cycle exist for $b > b_c$ or $b < b_c$? Explain, using the Poincaré-Benedixson theorem.

(d) Find the approximate period of the limit cycle for $b \simeq b_c$.

4. Suppose that $\dot{x} = f(x, \lambda)$ is a planar Hamiltonian system with parameter $\lambda \in \mathbb{R}$ (i.e. $x = (p, q)$, $f(p, q, \lambda) = \left(-\frac{\partial H}{\partial p}, \frac{\partial H}{\partial q}\right)$ for a Hamiltonian function $H(p, q, \lambda)$). Prove that such a system cannot have a saddle-node bifurcation.

   Hint: Recall that a saddle-node bifurcation requires that the zero eigenvalue of $D_x f(x_0, \lambda_0)$ (where $f(x_0, \lambda_0) = 0$) has algebraic multiplicity one.

5. Recall that a dynamical system $\dot{x} = f(x)$ on $\mathbb{R}^n$ is called chaotic it it possesses a compact invariant set $\Lambda \subset \mathbb{R}^n$ such the flow $\phi_t(x)$ is topologically transitive on Λ and has sensitive dependence on initial conditions on Λ. Consider the system given in polar coordinates by

$$\dot{r} = \sin\frac{\pi}{r},$$
$$\dot{\theta} = r$$

   (with the origin being a fixed point). Show that this system has orbits that exhibit sensitive dependence on initial conditions but is not chaotic. Proceed as follows:

   (a) Show that there are countably many periodic orbits given by

$$(r(t), \theta(t)) = \left(\frac{1}{n}, \frac{t}{n} + \theta_0\right), \quad n = 1, 2 \dots$$

   and verify that these orbits are stable for n even and unstable for n odd. Note that there is a compact invariant set containing countably many unstable periodic orbits.

   (b) Determine which orbits in open annuli bounded by adjacent stable periodic orbits exhibit sensitive dependence on initial conditions. Conclude that there is a compact invariant set on which the system has sensitive dependence on initial conditions.

   (c) Show that the system may only be topologically transitive in the open annuli bounded by adjacent stable and unstable periodic orbits.

   (d) Explain why the above steps imply that the system is not chaotic.

6. Suppose $\mathbf{A} \in \mathbb{R}^{m \times n}$.

(a) Show that the nullspace of $\mathbf{A}^T\mathbf{A}$ coincides with the nullspace of $\mathbf{A}$.

(b) Prove that $\mathbf{A}^T\mathbf{A}$ is invertible iff $\text{rank}\mathbf{A} = n$.