1. Consider the Lorenz system, which is a vastly oversimplified model of atmospheric convection:

\[
\begin{align*}
\dot{x} &= \sigma(y-x) \\
\dot{y} &= \rho x - y - xz \\
\dot{z} &= xy - \beta z
\end{align*}
\]

where \(\sigma, \rho, \beta \geq 0\).

(a) Suppose \(\rho < 1\). Show that the origin is a \textit{globally} asymptotically stable equilibrium.

\textit{Hint: Look for a Liapunov function in the form } \(ax^2 + by^2 + cz^2\)

(b) Show that regardless of parameter values, trajectories of the Lorenz system enter the ellipsoid

\[
\rho x^2 + \sigma y^2 + \sigma(z - 2\rho)^2 \leq c, \quad 0 < c < \infty
\]

within finite time and remain in it thereafter.

2. Consider a planar gradient system:

\[
\dot{x} = \nabla V(x) \quad x \in \mathbb{R}^2,
\]

where \(V: \mathbb{R}^2 \to \mathbb{R}\) is a smooth map. Prove that the non-wandering set of this system contains only fixed points and no periodic or homoclinic orbits are possible.

\textit{Hist: Use } \(V(x)\) \textit{as a Liapunov function.}

3. Consider the system

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= a(1 - x^4)y - x
\end{align*}
\]

(a) Find all equilibrium points, perform linear stability analysis, and classify the equilibrium points accordingly.

(b) Sketch the phase plane.

(c) Describe the bifurcation that occurs at \(a = 0\).

(d) Prove that there exists a unique closed orbit for this system when \(a > 0\).

(e) Show that all nonzero solutions of the system tend to this closed orbit when \(a > 0\).

4. Consider a discrete dynamical system \(x \mapsto g(x)\) where \(g: [0, 1] \to [0, 1]\) is defined by

\[
g(x) = \begin{cases} 
1 - 2x, & 0 \leq x \leq \frac{1}{2} \\
-1 + 2x, & \frac{1}{2} < x \leq 1
\end{cases}
\]
(a) Find fixed points of \( g \) and determine their stability.

(b) Determine how \( g \) acts on the binary expansion of any \( x \in [0, 1] \).

(c) Use binary expansions to show that \( g \) has points of any period and the set of all periodic points is dense in \([0, 1]\).

(d) Compute the Lyapunov exponent of \( g \).

5. Consider the vector \( x = \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} \). Construct a matrix \( A \in GL(3, \mathbb{R}) \) such that \( A \) satisfies all of the following:

(a) \( A^2 = I_3 \) (the identity matrix on \( \mathbb{R}^3 \))

(b) \( A \) has two positive eigenvalues.

(c) \( Ax = -x \)

6. A spherical ball of mochi with uniform temperature \( T_0 \), radius \( a \) and thermal diffusivity \( \kappa \) is thrown into icy water. Solve for the subsequent temperature \( T \) governed by the heat equation:

\[
\frac{\partial T}{\partial t} = \kappa \nabla^2 T,
\]

where the Laplacian is given in spherical coordinates:

\[
\nabla^2 = \frac{1}{r^2 \partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta \partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta \partial \varphi^2}
\]

Deduce that a sphere that is twice as large in diameter requires four times as long to cool down.