SAMPLE APPLIED MATH QUALIFYING EXAM 3

1. Consider the Lorenz system, which is a vastly oversimplified model of atmospheric convection:

$$\begin{aligned}\dot{x} &= \sigma(y - x)\\ \dot{y} &= \rho x - y - xz\\ \dot{z} &= xy - \beta z\end{aligned}$$

where $\sigma, \rho, \beta \geq 0$.

(a) Suppose $\rho < 1$. Show that the origin is a *globally* asymptotically stable equilibrium.
Hint: Look for a Liapunov function in the form $ax^2 + by^2 + cz^2$

(b) Show that regardless of parameter values, trajectories of the Lorenz system enter the ellipsoid

$$\rho x^2 + \sigma y^2 + \sigma(z - 2\rho)^2 \leq c, \quad 0 < c < \infty$$

within finite time and remain in it thereafter.

2. Consider a *planar* gradient system:

$$\dot{x} = \nabla V(x) \quad x \in \mathbb{R}^2,$$

where $V : \mathbb{R}^2 \to \mathbb{R}$ is a smooth map. Prove that the non-wandering set of this system contains only fixed points and no periodic or homoclinic orbits are possible.
Hist: Use $V(x)$ *as a Liapunov function.*

3. Consider the system

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= a(1 - x^4)y - x\end{aligned}$$

(a) Find all equilibrium points, perform linear stability analysis, and classify the equilibrium points accordingly.

(b) Sketch the phase plane.

(c) Describe the bifurcation that occurs at $a = 0$.

(d) Prove that there exists a unique closed orbit for this system when $a > 0$.

(e) Show that all nonzero solutions of the system tend to this closed orbit when $a > 0$.

4. Consider a discrete dynamical system $x \mapsto g(x)$ where $g : [0, 1] \to [0, 1]$ is defined by

$$g(x) = \begin{cases} 1 - 2x, & 0 \leq x \leq \dfrac{1}{2} \\ -1 + 2x, & \dfrac{1}{2} < x \leq 1 \end{cases}$$

(a) Find fixed points of g and determine their stability.

(b) Determine how g acts on the binary expansion of any $x \in [0, 1]$.

(c) Use binary expansions to show that g has points of any period and the set of all periodic points is dense in $[0, 1]$.

(d) Compute the Lyapunov exponent of g.

5. Consider the vector $x = \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$. Construct a matrix $A \in GL(3, \mathbb{R})$ such that A satisfies all of the following:

   (a) $A^2 = \mathrm{I}_3$ (the identity matrix on $\mathbb{R}^3$)

   (b) A has two positive eigenvalues.

   (c) $Ax = -x$

6. A spherical ball of mochi with uniform temperature T_0, radius a and thermal diffusivity κ is thrown into icy water. Solve for the subsequent temperature T governed by the heat equation:

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T,$$

where the Laplacian is given in spherical coordinates:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \varphi^2}$$

Deduce that a sphere that is twice as large in diameter requires four times as long to cool down.