SAMPLE APPLIED MATH QUALIFYING EXAM 4

1. Data from the Hudson Bay Company in Canada goes back to 1840 and shows periodic cycles in the populations of snowshoe hare (prey - P) and lynx (predator - Q).

   (a) Derive a model for these two populations, P and Q, under the assumptions:

      i. the prey population, P, grows exponentially in time if there are no predators,

      ii. the rate of predation, i.e., the rate at which the predators eat the prey, depends on the likelihood of encounters between the two populations, P and Q,

      iii. the growth rate of the predator population is proportional to food intake, i.e., proportional to the predation rate,

      iv. the predator population dies off exponentially in time if there are no prey.

   (b) Nondimensionalize the variables and parameters in these two equations and determine any remaining dimensionless parameter(s). Give an interpretation of the dimensionless parameter(s).

   (c) Determine the steady states of the system, and perform linear stability analysis and classify the steady states accordingly.

   (d) Sketch the (P, Q) phase plane including the nullclines and steady states.

   (e) Sketch the direction fields through the nullclines and of a few trajectories in the phase plane.

   (f) Determine the period of small amplitude oscillations about the steady state.

   (g) What can you say about the structural stability of this system?

2. Consider a discrete dynamical system defined by a C^r $(r \geq 1)$ diffeomorphism $x \mapsto g(x)$, $x \in \mathbb{R}^n$. Suppose that $x = \bar{x}$ is a nonwandering point, that is, for any neighborhood $U \ni \bar{x}$ there is $n \neq 0$ such that $g^n(U) \cap U \neq \emptyset$. Prove that there are countably many such n.

3. Consider the following variation of the predator-prey system:

$$\dot{x} = x(x(1-x) - y)$$
$$\dot{y} = y(x - a),$$

   where $x, y \geq 0$ represent prey and predator populations, respectively, and $a > 0$ is a control parameter.

   (a) Find equilibrium points, perform linear stability analysis, and classify the equilibrium points accordingly.

   (b) Deduce that the predators go extinct if $a > 1$

(c) Show that a Hopf bifurcation occurs at $a = \frac{1}{2}$.

4. Let $T = [0, 1] \times [0, 1]$ and consider a discrete dynamical system $x \mapsto g(x)$ where $g : T \to T$ is defined by

$$g(x, y) = (2x + y, x + y) \mod 1$$

Prove that a point $p \in T$ is periodic if and only if $p \in T \cap \mathbb{Q} \times \mathbb{Q}$, that is, p has rational coordinates.

5. Consider the following heat equation on the interval $[0, \pi]$:

$$u_t = u_{xx} \qquad x \in (0, \pi),\ t > 0,$$

$$u(0, t) = A \qquad u(\pi, t) = B, \qquad t > 0,$$

$$u(x, 0) = g(x), \qquad x \in (0, \pi),$$

where A, B are constants.

(a) Construct a formal solution.

(b) What conditions on $g(x)$ assure existence of a classical solution? Include a proof or state explicitly any theorems used.

(c) Find the equilibrium state of the temperature distribution.

6. Let U and V be subspaces of $\mathbb{R}^n$. Let P_U and P_V be $n \times n$ matrices that represent the orthogonal projections onto U and V, respectively. Let $W = U \cap V$. Prove that

$$S = \lim_{n \to \infty} (P_U P_V)^n$$

exists and that S represents the orthogonal projection onto W.