Hardy spaces of generalized analytic functions in the unit disc

Generalized analytic functions are defined as solutions, in the distributional sense of the following $\overline{\partial}$ equations

$$\overline{\partial} f = \nu \overline{\partial f} \tag{1}$$

or

$$\overline{\partial} w = \alpha \overline{w}, \text{ in } \mathbb{D} \tag{2}$$

where $\partial f = \partial_z f = \frac{1}{2}(\partial_x f - i\partial_y f)$ and $\overline{\partial} f = \partial_{\overline{z}} f = \frac{1}{2}(\partial_x f + i\partial_y f)$, $\alpha \in L^\infty(\mathbb{D})$ and $\nu \in W^{1,\infty}(\mathbb{D})$. For certain classes of coefficients α and ν, the two partial differential equations (1) and (2) are equivalent. Such functions have been introduced in 1954 [2]. They have been studied more recently because of their link with partial differential equations arising in mathematical physics [3]. We will start showing that a function solution of (1) satisfies generalized Cauchy-Riemann equations. Then, we will define the Hardy spaces of generalized analytic functions $H^p_\nu(\mathbb{D})$ (resp. $G^p_\alpha(\mathbb{D})$) for $1 < p < \infty$. Those spaces have been introduced in 2010 [1]. The space $H^p_\nu(\mathbb{D})$ (respectively $G^p_\alpha(\mathbb{D})$) is the collection of functions f (resp. w) solutions in the sense of distributions of Equation (1) (resp. Equation (2)) satisfying

$$\|f\|^p_{H^p_\nu(\mathbb{D})} := \operatorname*{ess\,sup}_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p dt < \infty. \tag{3}$$

Note that $H^p_0(\mathbb{D}) = H^p(\mathbb{D})$ and $G^p_0(\mathbb{D}) = H^p(\mathbb{D})$ where $H^p(\mathbb{D})$ denotes the classical Hardy space (collection of analytic functions on $\mathbb{D}$ satisfying (3)). For $\nu \in W^{1,\infty}(\mathbb{D})$ and $\alpha \in L^\infty(\mathbb{D})$ such that $\alpha = \frac{-\overline{\partial}\nu}{1-\nu^2}$, the two spaces $H^p_\nu(\mathbb{D})$ and $G^p_\alpha(\mathbb{D})$ are isomorphic. We will give some properties of $H^p_\nu(\mathbb{D})$-($G^p_\alpha(\mathbb{D})$ resp.)-functions and prove that they share many properties with $H^p(\mathbb{D})$-functions thanks to a factorization result of solutions of Equation (2).

In a second part, we will study some properties of composition operators C_ϕ on $H^p_\nu(\mathbb{D})$ and $G^p_\alpha(\mathbb{D})$ defined by $C_\phi(f) = f \circ \phi$. Such operators play an important role in operator theory. Indeed, they naturally appear in the changing of variables or through transformations between functional spaces. Moreover, there is a correspondance between properties of the operator C_ϕ and properties of the symbol ϕ.

References

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