THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

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The following results of the forty-second William Lowell Putnam Mathematical Competition, held on December 5, 1981, have been determined in accordance with the governing regulations. This annual contest is supported by the William Lowell Putnam Prize Fund for the Promotion of Scholarship, left by Mrs. Putnam in memory of her husband, and is held under the auspices of the Mathematical Association of America.

The first prize, five thousand dollars, was awarded to the Department of Mathematics of **Washington University**, St. Louis, Missouri. The members of its winning team were: Kevin P. Keating, Edward A. Shpiz, and Richard A. Stong; each was awarded a prize of two hundred fifty dollars.

The second prize, two thousand five hundred dollars, was awarded to the Department of Mathematics of **Princeton University**, Princeton, New Jersey. The members of its team were: Gregg N. Patruno, David P. Roberts, and Charles H. Walter; each was awarded a prize of two hundred dollars.

The third prize, one thousand five hundred dollars, was awarded to the Department of Mathematics of **Harvard University**, Cambridge, Massachusetts. The members of its team were: Michael J. Larsen, Laurence E. Penn, and Michael Raship; each was awarded a prize of one hundred fifty dollars.

The fourth prize, one thousand dollars, was awarded to the Department of Mathematics of **Stanford University**, Stanford, California. The members of its team were: Richard J. Beigel, Thomas C. Hales, and Kenneth O. Olum; each was awarded a prize of one hundred dollars.

The fifth prize, five hundred dollars, was awarded to the Department of Mathematics of the **University of Maryland**, College Park, Maryland. The members of its team were Ravi B. Boppana, Andrew E. Gelman, and Brian R. Hunt; each was awarded a prize of fifty dollars.

The five highest-ranking individual contestants, in alphabetical order, were **David W. Ash**, University of Waterloo; **Scott R. Fluhrer**, Case Western Reserve University; **Michael J. Larsen**, Harvard University; **Robin A. Pemantle**, University of California, Berkeley; and **Adam Stephanides**, University of Chicago. Each of these students was designated a Putnam Fellow by the Mathematical Association of America and awarded a prize of five hundred dollars by the Putnam Prize Fund.

The next five highest-ranking individuals, in alphabetical order, were *Benji N. Fisher*, Harvard University; *Brian R. Hunt*, University of Maryland, College Park; *Kevin P. Keating*, Washington University, St. Louis; *Kenneth O. Olum*, Stanford University; and *Richard A. Stong*, Washington University, St. Louis. Each of these students was awarded a prize of two hundred fifty dollars.

The following teams, named in alphabetical order, received honorable mention: *California Institute of Technology*, with team members Lance J. Dixon, Scott R. Johnson, Zinovy B. Reichstein; *Case Western Reserve University*, with team members Scott R. Fluhrer, Barak A. Pearlmutter, John D. Yeager; *University of Toronto*, with team members David M. Atwood, Ivo Klemes, Philip A. Mansfield; *University of Waterloo*, with team members David W. Ash, A. Michael O'Brien, Gordon J. Sinnamon; and *Yale University*, with team members Alan S. Edelman, Paul N. Feldman, Ronald H. Weinstock.

Honorable mention was achieved by the following thirty-four individuals, named in alphabetical order: *Troy W. Barbee*, Princeton University; *Richard J. Beigel*, Stanford University; *Gary M. Bernstein*, Princeton University; *Christopher J. Bishop*, Michigan State University; *Ravi B. Boppana*, University of Maryland, College Park; *Eric D. Carlson*, Michigan State University; *Stephen J. Curran*, Beloit College; *David L. Des Jardins*, Massachusetts Institute of Technology; *Robert A. Ewan*, Queen's University, Kingston; *Michael V. Finn*, Harvard University; *Nathaniel E. Glasser*, Yale University; *Lin Goldstein*, University of California, Berkeley; *Fred W. Helenius*, Massachusetts Institute of Technology; *Robert J. Holt*, Stanford University; *Irwin L. Jungreis*, Cornell University; *Joe J. Kilian*, Massachusetts Institute of Technology; *Ivo Klemes*, University of Toronto; *Mark K. Maginity*, The Johns Hopkins University; *Evan W. Morton*, Harvard University; *Gregg N. Patruno*, Princeton University; *Mark G. Pleszkoch*, University of Virginia; *Michael Raship*, Harvard University; *David P. Roberts*, Princeton University; *James R. Roche*, University of Notre Dame; *Daniel J. Scales*, Princeton University; *Brian F. Sheppard*, Harvard University; *Edward A. Shpiz*, Washington University, St. Louis; *Carlos T. Simpson*, Harvard University; *Bruce K. Smith*, Princeton University; *John M. Sullivan*, Harvard University; *Pierre Tremblay*, Université Laval; *Jerome V. Walsh*, University of Illinois, Urbana-Champaign; *Charles H. Walter*, Princeton University; *David I. Wolland*, Harvard University.

The other individuals who achieved ranks among the top 100, in alphabetical order of their schools, were: University of Alabama, *Richard B. Borie*; University of Alberta, *Michael P. Lamoureux*; Amherst College, *F. Miller Maley*; California Institute of Technology, *Bradley W. Brock*, *Lance J. Dixon*, *Scott R. Johnson*, *Forrest C. Quinn*, *Zinovy B. Reichstein*; University of California, Los Angeles, *Paul T. Lockhart*; University of California, Santa Barbara, *Eric S. Williams*; Carleton College, *Jon L. Shreve*; Case Western Reserve University, *Barak A. Pearlmutter*, *Susan G. Staples*; University of Chicago, *Daniel J. Goldstein*; Colorado State University, *Tom S. Watts*; Harvard University, *Theodore M. Alper*, *William I. Chang*, *Zachary M. Franco*, *David J. Montana*, *Laurence E. Penn*, *James G. Propp*, *Alfred D. Shapere*, *Gregory B. Sorkin*; Harvey Mudd College, *Matthew G. Hudelson*; Université Laval, *Frederic M. Gourdeau*; University of Maryland, College Park, *Andrew E. Gelman*; Massachusetts Institute of Technology, *Richard A. Shapiro*, *Joseph L. Shipman*; Michigan State University, *Karl A. Dahlke*, *Lloyd A. Rawley*; University of New Brunswick, *Christian Friesen*; State University of New York, Buffalo, *Pierre H. Abbat*; University of Pennsylvania, *Joel Mick*; Princeton University, *Mark P. Kleiman*, *Mark A. Prysant*, *Daniel S. Rokhsar*, *Stephen A. Vavisis*; Rensselaer Polytechnic Institute, *William John Harte*; Rice University, *Christopher T. Delevoryas*, *Joe D. Warren*; Stanford University, *Thomas C. Hales*; University of Toronto, *John J. Chew*, *John J. Im*; Vanderbilt University, *Hartwig P. Arenstorf*; Washington University, St. Louis, *Bard Bloom*, *Karl F. Narveson*; University of Washington, *Kin Y. Li*; University of Waterloo, *W. Ross Brown*, *Bev I. Cope*, *Mark N. Culp*, *Herbert J. Fichtner*, *A. Michael O'Brien*; University of Wisconsin, Madison, *Chris S. Jantzen*; Yale University, *Jeffrey W. Clark*, *Alan S. Edelman*, *Paul N. Feldman*.

There were 2043 individual contestants from 343 colleges and universities in Canada and the United States in the competition of December 5, 1981. Teams were entered by 251 institutions.

The Questions Committee for the forty-second competition consisted of K. B. Stolarsky (Chairman), W. J. Firey, and D. A. Hensley; they composed the problems listed below and were most prominent among those suggesting solutions.

PROBLEMS

Problem A-1

Let $E(n)$ denote the largest integer k such that 5^k is an integral divisor of the product $1^1 2^2 3^3 \cdots n^n$. Calculate

$$\lim_{n \to \infty} \frac{E(n)}{n^2}.$$

Problem A-2

Two distinct squares of the 8 by 8 chessboard C are said to be adjacent if they have a vertex or side in common. Also, g is called a C-gap if for every numbering of the squares of C with all the integers $1, 2, \ldots, 64$ there exist two adjacent squares whose numbers differ by at least g. Determine the largest C-gap g.

Problem A-3

Find

$$\lim_{t\to\infty}\left[e^{-t}\int_0^t\int_0^t \frac{e^x - e^y}{x - y}\,dx\,dy\right]$$

or show that the limit does not exist.

Problem A-4

A point P moves inside a unit square in a straight line at unit speed. When it meets a corner it escapes. When it meets an edge its line of motion is reflected so that the angle of incidence equals the angle of reflection.

Let $N(T)$ be the number of starting directions from a fixed interior point P_0 for which P escapes within T units of time. Find the least constant a for which constants b and c exist such that

$$N(T) \leq aT^2 + bT + c$$

for all $T > 0$ and all initial points P_0.

Problem A-5

Let $P(x)$ be a polynomial with real coefficients and form the polynomial

$$Q(x) = (x^2 + 1)P(x)P'(x) + x\left([P(x)]^2 + [P'(x)]^2\right).$$

Given that the equation $P(x) = 0$ has n distinct real roots exceeding 1, prove or disprove that the equation $Q(x) = 0$ has at least $2n - 1$ distinct real roots.

Problem A-6

Suppose that each of the vertices of ΔABC is a lattice point in the (x, y)-plane and that there is exactly one lattice point P in the *interior* of the triangle. The line AP is extended to meet BC at E. Determine the largest possible value for the ratio of lengths of segments

$$\frac{|AP|}{|PE|}.$$

[A lattice point is a point whose coordinates x and y are integers.]

Problem B-1

Find

$$\lim_{n\to\infty}\left[\frac{1}{n^5}\sum_{h=1}^{n}\sum_{k=1}^{n}(5h^4 - 18h^2k^2 + 5k^4)\right].$$

Problem B-2

Determine the minimum value of

$$(r-1)^2 + \left(\frac{s}{r} - 1\right)^2 + \left(\frac{t}{s} - 1\right)^2 + \left(\frac{4}{t} - 1\right)^2$$

for all real numbers r, s, t with $1 \leq r \leq s \leq t \leq 4$.

Problem B-3

Prove that there are infinitely many positive integers n with the property that if p is a prime divisor of $n^2 + 3$, then p is also a divisor of $k^2 + 3$ for some integer k with $k^2 < n$.

Problem B-4

Let V be a set of 5 by 7 matrices, with real entries and with the property that $rA + sB \in V$ whenever $A, B \in V$ and r and s are scalars (i.e., real numbers). *Prove or disprove* the following assertion: If V contains matrices of ranks 0, 1, 2, 4, and 5, then it also contains a matrix of rank 3.

[The rank of a nonzero matrix M is the largest k such that the entries of some k rows and some k columns form a k by k matrix with a nonzero determinant.]

Problem B-5

Let $B(n)$ be the number of ones in the base two expression for the positive integer n. For example, $B(6) = B(110_2) = 2$ and $B(15) = B(1111_2) = 4$. Determine whether or not

$$\exp\left(\sum_{n=1}^{\infty} \frac{B(n)}{n(n+1)}\right)$$

is a rational number. Here $\exp(x)$ denotes e^x.

Problem B-6

Let C be a fixed unit circle in the Cartesian plane. For any convex polygon P each of whose sides is tangent to C, let $N(P, h, k)$ be the number of points common to P and the unit circle with center at (h, k). Let $H(P)$ be the region of all points (x, y) for which $N(P, x, y) \geq 1$ and $F(P)$ be the area of $H(P)$. Find the smallest number u with

$$\frac{1}{F(P)} \iint N(P, x, y)\, dx\, dy < u$$

for all polygons P, where the double integral is taken over $H(P)$.

In the 12-tuple $(n_{10}, n_9, \ldots, n_0, n_{-1})$ following each problem number below, n_i for $10 \geq i \geq 0$ is the number of students among the top 209 contestants achieving i points for the problem and n_{-1} is the number of those not submitting solutions.

SOLUTIONS

A-1. (33, 16, 13, 6, 0, 0, 0, 5, 23, 65, 39, 9)

We show that the limit is $1/8$. Let $T(m) = 1 + 2 + \cdots + m = m(m+1)/2$, $[x]$ denote the greatest integer in x, $h = [\log_5 n]$, and e_i be the fractional part $(n/5^i) - [n/5^i]$ for $1 \leq i \leq h$. Then

$$E(n) = 5T([n/5]) + 5^2 T([n/5^2]) + \cdots + 5^h T([n/5^h])$$

$$2E(n) = 5([n/5]^2 + [n/5]) + 5^2([n/5^2]^2 + [n/5^2]) + \cdots + 5^h([n/5^h]^2 + [n/5^h])$$

$$= 5\left(\frac{n^2}{5^2} - \frac{2e_1 n}{5} + e_1^2 + \frac{n}{5} - e_1\right) + \cdots + 5^h\left(\frac{n^2}{5^{2h}} - \frac{2e_h n}{5^h} + e_h^2 + \frac{n}{5^h} - e_h\right)$$

$$\frac{E(n)}{n^2} = \frac{1}{2}\left(\frac{1}{5} + \frac{1}{5^2} + \cdots + \frac{1}{5^h}\right) + \frac{h}{2n} - \frac{e_1 + e_2 + \cdots + e_h}{n}$$

$$+ \frac{5(e_1^2 - e_1) + \cdots + 5^h(e_h^2 - e_h)}{2n^2}.$$

Since $5^h \leq n < 5^{h+1}$ and $0 \leq e_i < 1$, one sees that $h/n \to 0$ and $E(n)/n^2 \to 1/8$ as $n \to \infty$.

A-2. (86, 17, 4, 0, 1, 1, 0, 0, 28, 2, 24, 46)

For any numbering, one can go from the square numbered 1 to the square numbered 64 in 7 or fewer steps, in each step going to an adjacent square; thus $(64 - 1)/7 = 9$ is a C-gap. It is the largest C-gap since with coordinates (a, b), $1 \leq a \leq 8$ and $1 \leq b \leq 8$, for the squares we can number (a, b) with $8(a - 1) + b$ and thus find that no number greater than 9 is a C-gap.

A-3. (1, 0, 5, 0, 0, 0, 0, 0, 4, 14, 64, 121)

Let $G(t)$ be the double integral. Then

$$\lim_{t\to\infty}\left[G(t)/e^t\right] = \lim_{t\to\infty}\left[G'(t)/e^t\right]$$

by L'Hôpital's Rule. One finds that

$$G'(t) = \int_0^t \frac{e^x - e^t}{x - t}\,dx + \int_0^t \frac{e^y - e^t}{y - t}\,dy = 2\int_0^t \frac{e^x - e^t}{x - t}\,dx.$$

Then using $e^x = e^t[1 + (x - t) + (x - t)^2/2! + \cdots]$, one sees that $e^{-t}G'(t) \to \infty$ as $t \to \infty$ since for sufficiently large t,

$$\frac{G'(t)}{2e^t} = \int_0^t \frac{e^{x-t} - 1}{x - t}\,dx = \int_0^t \frac{1 - e^{-y}}{y}\,dy > \int_1^t \frac{1 - e^{-y}}{y}\,dy > (1 - e^{-1})\log t.$$

A-4. (1, 10, 9, 0, 0, 0, 0, 20, 37, 17, 21, 94)

Set up coordinates so that a vertex of the given unit square is $(0, 0)$ and two sides of the square are on the axes. Using the reflection properties, one can see that P escapes within T units of time if and only if the (infinite) ray from P_0, with the direction of the first segment of the path, goes through a lattice point (point with integer coordinates) within T units of distance from P_0. Thus $N(T)$ is at most the number $L(T)$ of lattice points in the circle with center at P_0 and radius T. Tiling the plane with unit squares having centers at the lattice points and considering areas, one sees that

$$N(T) \le L(T) \le \pi\left[T + (\sqrt{2}/2)\right]^2.$$

Hence there is an upper bound for $N(T)$ of the form $\pi T^2 + bT + c$, with b and c fixed. When just one coordinate of P_0 is irrational,

$$N(T) = L(T) \ge \pi\left[T - (\sqrt{2}/2)\right]^2.$$

This lower bound for $N(T)$ exceeds $aT^2 + bT + c$ for sufficiently large T if $a < \pi$; hence π is the desired a.

A-5. (3, 3, 9, 0, 0, 0, 0, 0, 10, 6, 49, 129)

We show that $Q(x)$ has at least $2n - 1$ real zeros. One finds that $Q(x) = F(x)G(x)$, where

$$F(x) = P'(x) + xP(x) = e^{-x^2/2}\left[e^{x^2/2}P(x)\right]', \quad G(x) = xP'(x) + P(x) = [xP(x)]'.$$

We can assume that $P(x)$ has exactly n zeros a_i exceeding 1 with $1 < a_1 < a_2 < \cdots < a_n$. It follows from Rolle's Theorem that $F(x)$ has $n - 1$ zeros b_i and $G(x)$ has n zeros c_i with

$$1 < a_1 < b_1 < a_2 < b_2 < \cdots < b_{n-1} < a_n, \quad 0 < c_1 < a_1 < c_2 < a_2 < \cdots < c_n < a_n.$$

If $b_i \neq c_{i+1}$ for all i, the b's and c's are $2n - 1$ distinct zeros of $Q(x)$. So we assume that $b_i = c_{i+1} = r$ for some i. Then

$$P'(r) + rP(r) = 0 = rP'(r) + P(r)$$

and so $(r^2 - 1)P(r) = 0$. Since $r = b_i > 1$, $P(r) = 0$. Since $a_i < r < a_{i+1}$, this contradicts the fact that the a's are all the zeros exceeding 1 of $P(x)$. Hence $Q(x)$ has at least $2n - 1$ distinct real zeros.

A-6. (1, 0, 1, 1, 0, 0, 0, 0, 17, 9, 45, 135)

Treating each point X of the plane as the vector $\overrightarrow{AX}$ with initial point at A and final point at X, let

$$L = (B + C)/2, M = C/2, \quad \text{and} \quad N = B/2$$

(be the midpoints of sides BC, AC, and AB). Also let

$$S = (2L + M)/3 = (B + C + M)/3, T = (2L + N)/3$$
$$= (B + C + N)/3, Q = 2P - B, \quad \text{and} \quad R = 3P - B - C.$$

Clearly Q and R are lattice points. Also $Q \neq P$ and $R \neq P$ since $Q = P$ implies $P = B$ and $R = P$ implies that P is the point L on side BC. Hence Q is not inside ΔABC and this implies that P is not inside ΔNBL since the linear transformation f with $f(X) = 2X - B$ translates a doubled ΔNBL (and its inside) onto ΔABC (and its inside). Similarly, P is not inside ΔMCL. Using the mapping $g(X) = 3X - B - C$ and the fact that R is not inside ΔLMN, one finds that P is not inside ΔLST. Since the distance from A to line ST is 5 times the distance between lines ST and BC, it follows that $|AP|/|PE| \leq 5$. This upper bound 5 is seen to be the maximum by considering the example with $A = (0,0)$, $B = (0,2)$, and $C = (3,0)$ in which $P = (1,1) = T$ is the only lattice point inside ΔABC and $|AT|/|TE| = 5$.

B-1. (78, 19, 33, 0, 5, 0, 0, 0, 15, 11, 33, 15)

Let $S_k(n) = 1^k + 2^k + \cdots + n^k$. Using standard methods of calculus texts one finds that

$$S_2(n) = (n^3/3) + (n^2/2) + an$$

and

$$S_4(n) = (n^5/5) + (n^4/2) + bn^3 + cn^2 + dn,$$

with a, b, c, d constants. Then the double sum is

$$10nS_4(n) - 18[S_2(n)]^2 = (2n^6 + 5n^5 + \cdots) - (2n^6 + 6n^5 + \cdots) = -n^5 + \cdots$$

and the desired limit is -1.

B-2. (37, 1, 10, 0, 0, 0, 0, 0, 6, 9, 123, 23)

First we let $0 < a < b$ and seek the x that minimizes

$$f(x) = \left(\frac{x}{a} - 1\right)^2 + \left(\frac{b}{x} - 1\right)^2 \text{ on } a \leq x \leq b.$$

Let $x/a = z$ and $b/a = c$. Then

$$f(x) = g(z) = (z - 1)^2 + \left(\frac{c}{z} - 1\right)^2.$$

Now $g'(z) = 0$ implies

$$z^4 - z^3 + cz - c^2 = (z^2 - c)(z^2 - z + c) = 0;$$

the only positive solution is $z = \sqrt{c}$. Since $0 < a < b$, $c > 1, \sqrt{c} > 1$, and

$$g(1) = g(c) = (c - 1)^2 = (\sqrt{c} - 1)^2(\sqrt{c} + 1)^2 > 2(\sqrt{c} - 1)^2 = g(\sqrt{c}).$$

Hence the minimum of $g(z)$ on $1 \leq z \leq c$ occurs at $z = \sqrt{c}$. It follows that the minimum for $f(x)$ on $a \leq x \leq b$ occurs at $x = a\sqrt{b/a} = \sqrt{ab}$. Then the minimum for the given function of r, s, t occurs with $r = \sqrt{s}$, $t = \sqrt{4s} = 2r$, and $s = \sqrt{rt} = r\sqrt{2}$. These imply that $r = \sqrt{2}$, $s = 2$, $t = 2\sqrt{2}$. Thus the desired minimum value is $4(\sqrt{2} - 1)^2 = 12 - 8\sqrt{2}$.

B-3. (5, 4, 1, 0, 0, 0, 0, 0, 3, 0, 56, 140)

As m ranges through all nonnegative integers,

$$n = (m^2 + m + 2)(m^2 + m + 3) + 3$$

takes on an infinite set of positive integral values. Let $f(x) = x^2 + 3$. Examination of $\{f(m)\} = 3, 4, 7, 12, 17, 28, 39, 52, 67, 84, \ldots$ leads one to conjecture that

$$f(x)f(x+1) = f[x(x+1)+3] = f(x^2+x+3).$$

This is easily proved. Using this property and the above relation between m and n, one has

$$f(n) = f(m^2+m+2)f(m^2+m+3) = f(m^2+m+2)f(m)f(m+1).$$

Thus $p|f(n)$ with p prime implies that $p|f(k)$ with k equal to m, $m+1$, or m^2+m+2. Since each of these possibilities for k satisfies $k^2 < n$, the desired result follows.

B-4. (38, 4, 3, 1, 0, 0, 0, 1, 3, 6, 28, 125)

Let $M = M(a, b, c)$ denote the 5 by 7 matrix (a_{ij}) with

$$a_{11} = a,\ a_{22} = a_{33} = a_{44} = a_{55} = b,\ a_{16} = a_{27} = c,$$

and $a_{ij} = 0$ in all other cases. Then the set V of all such M (with a, b, c arbitrary real numbers) is closed under linear combinations. Also, $M(0,0,0)$, $M(1,0,0)$, $M(0,0,1)$, $M(0,1,0)$, and $M(1,1,0)$ have ranks 0, 1, 2, 4, and 5, respectively. But no M in V has rank 3 since $b \neq 0$ implies that the rank is 4 or 5 and $b = 0$ forces the rank to be 0, 1, or 2.

B-5. (7, 10, 7, 6, 3, 1, 1, 3, 2, 6, 49, 114)

If n has d digits in base 2, $2^{d-1} \le n$ and so

$$B(n) \le d \le 1 + \log_2 n.$$

This readily implies that $\Sigma_{n=1}^{\infty}[B(n)/n(n+1)]$ converges to a real number S. Hence the manipulations below with convergent series are allowable in the two solutions which follow.

Each n is uniquely expressible as $n_0 + 2n_1 + 2^2 n_2 + \cdots$ with each n_i in $\{0, 1\}$ (and with $n_i = 0$ for all but a finite set of i). Since

$$1 + 2 + 2^2 + \cdots + 2^{i-1} = 2^i - 1,$$

one sees that $n_i = 1$ if and only if n is of the form $k + 2^i + 2^{i+1}j$ with k in $\{0, 1, \ldots, 2^i - 1\}$ and j in $\{0, 1, 2, \ldots\}$. Thus

$$S = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \sum_{i=0}^{\infty} n_i$$
$$= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{2^i-1} \frac{1}{(k + 2^i + 2^{i+1}j)(1 + k + 2^i + 2^{i+1}j)}.$$

Using $1/s(s+1) = 1/s - 1/(s+1)$, the innermost sum telescopes and

$$S = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left[\frac{1}{2^i(1+2j)} - \frac{1}{2^i(2+2j)}\right] = \sum_{i=0}^{\infty} \frac{1}{2^i} \sum_{j=0}^{\infty} (-1)^j \frac{1}{j}.$$

Since it is well known that $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2$,

$$S = \left(\sum_{i=0}^{\infty} 2^{-i}\right) \ln 2 = 2\ln 2 = \ln 4$$

and e^S is the rational number 4.

Alternatively, we note that $B(2m) = B(m)$, $B(2m+1) = 1 + B(2m) = 1 + B(m)$.

Then

$$S = \sum_{n=1}^{\infty} \frac{B(n)}{n(n+1)} = \sum_{m=0}^{\infty} \frac{B(2m+1)}{(2m+1)(2m+2)} + \sum_{m=1}^{\infty} \frac{B(2m)}{2m(2m+1)}$$

$$= \sum_{m=0}^{\infty} \frac{1 + B(m)}{(2m+1)(2m+2)} + \sum_{m=1}^{\infty} \frac{B(m)}{2m(2m+1)}$$

$$= \sum_{m=0}^{\infty} \frac{1}{(2m+1)(2m+2)} + \sum_{m=1}^{\infty} B(m)\left[\frac{1}{2m(2m+1)} + \frac{1}{(2m+1)(2m+2)}\right]$$

$$= \ln 2 + \frac{1}{2} \sum_{m=1}^{\infty} \frac{B(m)}{m(m+1)} = \ln 2 + \frac{S}{2}.$$

Hence $S/2 = \ln 2$, $S = \ln 4$, and $\exp(S)$ is the rational number 4.

B-6. (10, 0, 2, 0, 0, 0, 0, 0, 0, 7, 52, 138)

Let $L = L(P)$ be the perimeter of P. One sees that $H(P)$ consists of the region bounded by P, the regions bounded by rectangles whose bases are the sides of P and whose altitudes equal 1, and sectors of unit circles which can be put together to form one unit circle. Hence

$$F(P) = (L/2) + L + \pi = \pi + 3L/2.$$

If A and B are two consecutive vertices of P, the contribution of side AB to the double integral I is double the area of the region (of the figure) bounded by the unit semicircles with centers at A and B and segments CD and EF such that $ABCD$ and $ABEF$ are rectangles and $|AD| = 1 = |AF|$.

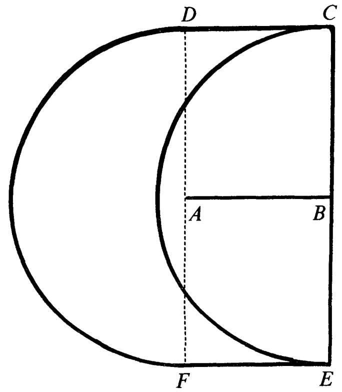

FIG. 1.

One doubles this area because there is a symmetric region bounded by CD, EF, and the other halves of the unit circles centered at A and B. The overlap of the two regions counts twice. By Cavalieri's slicing principle, this contribution of side AB to I is 4 times the length of AB. Hence $I = 4L$ and

$$\frac{I}{F(P)} = \frac{4L}{\pi + 3L/2} = \frac{8}{3 + (2\pi/L)}.$$

One can make L arbitrarily large (e.g., by letting P be a triangle with two angles arbitrarily close to right angles). Hence the desired least upper bound is $8/3$.