



## The William Lowell Putnam Mathematical Competition

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## THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

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The following results of the forty-third William Lowell Putnam Mathematical Competition, held on December 4, 1982, have been determined in accordance with the governing regulations. This annual contest is supported by the William Lowell Putnam Prize Fund for the Promotion of Scholarship, left by Mrs. Putnam in memory of her husband, and is held under the auspices of the Mathematical Association of America.

The first prize, five thousand dollars, was awarded to the Department of Mathematics of **Harvard University**, Cambridge, Massachusetts. The members of its winning team were: Benji N. Fisher, Michael J. Larsen, and Michael Raship; each was awarded a prize of two hundred fifty dollars.

The second prize, two thousand five hundred dollars, was awarded to the Department of Mathematics of the **University of Waterloo**, Waterloo, Ontario, Canada. The members of its team were: David W. Ash, W. Ross Brown, and Herbert J. Fichtner; each was awarded a prize of two hundred dollars.

The third prize, one thousand five hundred dollars, was awarded to the Department of Mathematics of the **California Institute of Technology**, Pasadena, California. The members of its team were: Bradley W. Brock, Scott R. Johnson, and Zinovy B. Reichstein; each was awarded a prize of one hundred fifty dollars.

The fourth prize, one thousand dollars, was awarded to the Department of Mathematics of **Yale University**, New Haven, Connecticut. The members of its team were Alan S. Edelman, Paul N. Feldman, and Nathaniel E. Glasser; each was awarded a prize of one hundred dollars.

The fifth prize, five hundred dollars, was awarded to the Department of Mathematics of **Princeton University**, Princeton, New Jersey. The members of its team were Gregg N. Patrino, David P. Roberts, and Daniel J. Scales; each was awarded a prize of fifty dollars.

The five highest-ranking individual contestants, in alphabetical order, were **David W. Ash**, University of Waterloo; **Eric D. Carlson**, Michigan State University; **Noam D. Elkies**, Columbia University; **Brian R. Hunt**, University of Maryland, College Park; and **Edward A. Shpiz**, Washington University, St. Louis. Each of these students was designated a Putnam Fellow by the Mathematical Association of America and awarded a prize of five hundred dollars by the Putnam Prize Fund.

The next six highest-ranking individuals, in alphabetical order, were *Joel Friedman*, Harvard University; *Irwin L. Jungreis*, Cornell University; *Michael J. Larsen*, Harvard University; *Gregg N. Patrino*, Princeton University; *Robin A. Pemantle*, University of California, Berkeley; and *Mark G. Pleszkoch*, University of Virginia. Each of these students was awarded a prize of two hundred fifty dollars.

The following teams, named in alphabetical order, received honorable mention: *Brown University*, with team members Stephen A. DiPippo, Edward F. Grove, and Erik R. Paulsen; *University of California, Berkeley*, with team members John W. Jones, Robin A. Pemantle, and Richard W. Webb; *Michigan State University*, with team members Eric D. Carlson, Erin J. Schram, and Frank J. Sottile; *University of Toronto*, with team members John J. Chew, John J. Im, Alastair M. Rucklidge; and *Washington University*, St. Louis, with team members Bard Bloom, Edward A. Shpiz, and Richard A. Stong.

Honorable mention was achieved by the following thirty-three individuals, named in alphabetical order: *Gary M. Bernstein*, Princeton University; *Bradley W. Brock*, California Institute of Technology; *W. Ross Brown*, University of Waterloo; *Bev I. Cope*, University of Waterloo; *Charles J. Cuny*, California Institute of Technology; *Michael J. Dawson*, Massachusetts Institute of Technology; *David L. Desjardins*, Massachusetts Institute of Technology; *Stephen A. DiPippo*, Brown University; *Paul N. Feldman*, Yale University; *Michael V. Finn*, Harvard University; *Zachary M. Franco*, Harvard University; *Tom Ilmanen*, Haverford College; *John J. Im*, University of Toronto; *Scott R. Johnson*, California Institute of Technology; *Kin Y. Li*, University of Washington; *F. Miller Maley*, Amherst College; *Yair N. Minsky*, Princeton University; *Evan W. Morton*, Harvard University; *Alan G. Murray*, California Institute of Technology; *Laurence E. Penn*, Harvard University; *Jeremy D. Primer*, Princeton University; *Michael Raship*, Harvard University; *David P. Roberts*, Princeton University; *James R. Roche*, University of Notre Dame; *Richard A. Shapiro*, Massachusetts Institute of Technology; *Carlos T. Simpson*, Harvard University; *Arthur P. Smith*, Memorial University of Newfoundland; *Thomas R. Stevenson*, University of British Columbia; *Richard A. Stong*, Washington University, St. Louis; *John M. Sullivan*, Harvard University; *Jerome V. Walsh*, University of Illinois at Urbana-Champaign; *Dwight S. Wilson*, Johns Hopkins University; and *David Wolland*, Harvard University.

The other individuals who achieved ranks among the top 100, in alphabetical order of their schools, were: University of Alberta, *Arthur B. Baragar*, *Robert P. Morewood*; University of Arizona, *Jan M. O. Soderkvist*; Biola University, *Mark M. Shimozone*; University of British Columbia, *Edmond D. Chow*; Brown University, *Erik R. Paulsen*; California Institute of Technology, *Pang-Chieh Chen*, *R. Sekhar Chivukula*, *James T. Lin*, *Vladimir S. Matijasevic*, *Zinovy B. Reichstein*; University of California at Santa Barbara, *John R. Kelly*; Carleton University, *Mike D. Dixon*; Case Western Reserve University, *Kevin E. Kelso*; University of Chicago, *Keith A. Ramsay*, *Yongbum Park*, *Michael P. Spertus*, *David S. Yuen*; University of Colorado, Boulder, *Boris Lerner*; Harvard University, *Frederick R. Adler*, *Bruce W. K. Brandt*, *Benji N. Fisher*, *Howard M. Pollack*, *Gregory B. Sorkin*; Haverford College, *Samuel R. Evens*; Iowa State University, *William R. Somskey*; Université Laval, *Line Baribeau*; University of Maryland, College Park, *Dougin A. Walker*; Massachusetts Institute of Technology, *David E. Brahm*, *Andrew E. Gelman*, *Daniel S. Lewart*, *Tomasz S. Mrowka*; Michigan State University, *Frank J. Sottile*; University of Michigan, Ann Arbor, *David A. Short*; University of Michigan, Dearborn, *Gregory T. Parker*; University of New Brunswick, *Christian Friesen*; North Carolina State University, *Samuel P. White*; Pomona College, *Alan M. Nadel*; Princeton University, *Kazuko Suzuki*, *Burt J. Totaro*, *Kevin M. Walker*; Queen's College of the City University of New York, *Boris Aronov*; Reed College, *Paul S. Hsieh*; Rensselaer Polytechnic Institute, *Benjamin J. Patz*, *William J. Harte*; Rose-Hulman Institute of Technology, *Randy L. Ekl*; Rutgers University, *Scott E. Axelrod*; Simon Fraser University, *David G. Wagner*; Sioux Falls College, *William H. Paulsen*; Stanford University, *Cris G. Poor*; University of Toronto, *John J. Chew*, *Peter A. Miegom*; Washington University, St. Louis, *Patrick J. Abegg*, *Paul H. Burchard*; University of Waterloo, *Herbert S. Fichtner*, *Charles R. Smith*, *Charles S. A. Timar*; University of Wisconsin, *Jon G. Udell*; Yale University, *Alan S. Edelman*, *Nathaniel E. Glasser*.

There were 2024 individual contestants from 348 colleges and universities in Canada and the United States in the competition of December 4, 1982. Teams were entered by 249 institutions.

The Questions Committee for the forty-third competition consisted of William J. Firey (Chairman), Douglas A. Hensley, and Melvin Hochster; they composed the problems listed below and were most prominent among those suggesting solutions.

## PROBLEMS

### Problem A-1

Let  $V$  be the region in the cartesian plane consisting of all points  $(x, y)$  satisfying the simultaneous conditions

$$|x| \leq y \leq |x| + 3 \quad \text{and} \quad y \leq 4.$$

Find the centroid  $(\bar{x}, \bar{y})$  of  $V$ .

**Problem A-2**

For positive real  $x$ , let

$$B_n(x) = 1^x + 2^x + 3^x + \cdots + n^x.$$

Prove or disprove the convergence of

$$\sum_{n=2}^{\infty} \frac{B_n(\log_n 2)}{(n \log_2 n)^2}.$$

**Problem A-3**

Evaluate

$$\int_0^{\infty} \frac{\text{Arctan}(\pi x) - \text{Arctan } x}{x} dx.$$

**Problem A-4**

Assume that the system of simultaneous differential equations

$$y' = -z^3, \quad z' = y^3$$

with the initial conditions  $y(0) = 1, z(0) = 0$  has a unique solution  $y = f(x), z = g(x)$  defined for all real  $x$ . Prove that there exists a positive constant  $L$  such that for all real  $x$ ,

$$f(x + L) = f(x), \quad g(x + L) = g(x).$$

**Problem A-5**

Let  $a, b, c$ , and  $d$  be positive integers and

$$r = 1 - \frac{a}{b} - \frac{c}{d}.$$

Given that  $a + c \leq 1982$  and  $r > 0$ , prove that

$$r > \frac{1}{1983^3}.$$

**Problem A-6**

Let  $\sigma$  be a bijection of the positive integers, that is, a one-to-one function from  $\{1, 2, 3, \dots\}$  onto itself. Let  $x_1, x_2, x_3, \dots$  be a sequence of real numbers with the following three properties:

- (i)  $|x_n|$  is a strictly decreasing function of  $n$ ;
- (ii)  $|\sigma(n) - n| \cdot |x_n| \rightarrow 0$  as  $n \rightarrow \infty$ ;
- (iii)  $\lim_{n \rightarrow \infty} \sum_{k=1}^n x_k = 1$ .

Prove or disprove that these conditions imply that

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n x_{\sigma(k)} = 1.$$

**Problem B-1**

Let  $M$  be the midpoint of side  $BC$  of a general  $\triangle ABC$ . Using the *smallest possible*  $n$ , describe a method for cutting  $\triangle AMB$  into  $n$  triangles which can be reassembled to form a triangle congruent to  $\triangle AMC$ .

**Problem B-2**

Let  $A(x, y)$  denote the number of points  $(m, n)$  in the plane with integer coordinates  $m$  and  $n$  satisfying

$m^2 + n^2 \leq x^2 + y^2$ . Let  $g = \sum_{k=0}^{\infty} e^{-k^2}$ . Express

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A(x, y) e^{-x^2 - y^2} dx dy$$

as a polynomial in  $g$ .

### Problem B-3

Let  $p_n$  be the probability that  $c + d$  is a perfect square when the integers  $c$  and  $d$  are selected independently at random from the set  $\{1, 2, 3, \dots, n\}$ . Show that  $\lim_{n \rightarrow \infty} (p_n \sqrt{n})$  exists and express this limit in the form  $r(\sqrt{s} - t)$ , where  $s$  and  $t$  are integers and  $r$  is a rational number.

### Problem B-4

Let  $n_1, n_2, \dots, n_s$  be distinct integers such that

$$(n_1 + k)(n_2 + k) \cdots (n_s + k)$$

is an integral multiple of  $n_1 n_2 \cdots n_s$  for every integer  $k$ . For each of the following assertions, give a proof or a counterexample:

- (a)  $|n_i| = 1$  for some  $i$ .  
 (b) If further all  $n_i$  are positive, then

$$\{n_1, n_2, \dots, n_s\} = \{1, 2, \dots, s\}.$$

### Problem B-5

For each  $x > e^e$  define a sequence  $S_x = u_0, u_1, u_2, \dots$  recursively as follows:  $u_0 = e$ , while for  $n \geq 0$ ,  $u_{n+1}$  is the logarithm of  $x$  to the base  $u_n$ . Prove that  $S_x$  converges to a number  $g(x)$  and that the function  $g$  defined in this way is continuous for  $x > e^e$ .

### Problem B-6

Let  $K(x, y, z)$  denote the area of a triangle whose sides have lengths  $x, y$ , and  $z$ . For any two triangles with sides  $a, b, c$  and  $a', b', c'$ , respectively, prove that

$$\sqrt{K(a, b, c)} + \sqrt{K(a', b', c')} \leq \sqrt{K(a + a', b + b', c + c')}$$

and determine the cases of equality.

## SOLUTIONS

In the 12-tuples  $(n_{10}, n_9, \dots, n_0, n_{-1})$  following each problem number below,  $n_i$  for  $10 \geq i \geq 0$  is the number of students among the top 201 contestants achieving  $i$  points for the problem and  $n_{-1}$  is the number of those not submitting solutions.

**A-1.** (146, 9, 2, 5, 0, 0, 0, 5, 2, 9, 22, 1)

Let  $T$  consist of the points inside or on the triangle with vertices at  $(0, 3), (-1, 4), (1, 4)$  and let  $U$  be the set of points inside or on the triangle with vertices at  $(0, 0), (-4, 4), (4, 4)$ . Then  $T$  and  $U$  overlap only on boundary points and their union is  $U$ . The centroids of  $T$  and  $U$  are  $(0, 11/3)$  and  $(0, 8/3)$ , respectively. The areas of  $T, U$ , and  $U$  are 1, 15, and 16, respectively. Using weighted averages with the areas as weights, one has

$$1 \cdot 0 + 15\bar{x} = 16 \cdot 0, \quad 1 \cdot \frac{11}{3} + 15\bar{y} = 16 \cdot \frac{8}{3}.$$

It follows that  $\bar{x} = 0, \bar{y} = 13/5$ .

**A-2.** (68, 17, 18, 7, 0, 3, 1, 0, 10, 2, 37, 38)

Since  $x = \log_n 2 > 0$ ,  $B_n(x) = 1^x + 2^x + \cdots + n^x \leq n \cdot n^x$  and

$$0 \leq \frac{B_n(\log_n 2)}{(n \log_2 n)^2} \leq \frac{n \cdot n^{\log_n 2}}{(n \log_2 n)^2} = \frac{2}{n(\log_2 n)^2}.$$

As  $\sum_{n=2}^{\infty} [2/n(\log_2 n)^2]$  converges by the integral test, the given series converges by the comparison test.

A-3. (14, 19, 11, 5, 0, 0, 0, 2, 5, 0, 37, 108)

$$\begin{aligned} \int_0^{\infty} \frac{\text{Arctan}(\pi x) - \text{Arctan } x}{x} dx &= \int_0^{\infty} \frac{1}{x} \text{Arctan}(ux) \Big|_{u=1}^{u=\pi} dx \\ &= \int_0^{\infty} \int_1^{\pi} \frac{1}{1+(xu)^2} du dx = \int_1^{\pi} \int_0^{\infty} \frac{1}{1+(xu)^2} dx du \\ &= \int_1^{\pi} \frac{1}{u} \cdot \frac{\pi}{2} du = \frac{\pi}{2} \ln \pi. \end{aligned}$$

A-4. (0, 2, 4, 5, 0, 0, 0, 11, 9, 22, 33, 115)

The differential equations imply that

$$y^3 y' + z^3 z' = z' y' - y' z' = 0$$

and hence that  $y^4 + z^4$  is constant. This and the initial conditions give  $y^4 + z^4 = 1$ . Thinking of  $x$  as a time variable and  $(y, z)$  as the coordinates of a point in a plane, this point moves on the curve  $y^4 + z^4 = 1$  with speed

$$[(y')^2 + (z')^2]^{1/2} = (z^6 + y^6)^{1/2}.$$

At any time, either  $y^4$  or  $z^4$  is at least  $\frac{1}{2}$  and so the speed is at least  $\{[(\frac{1}{2})^{1/4}]^6\}^{1/2}$ . Hence the point will go completely around the finite curve in some time  $L$ . As the speed depends only on  $y$  and  $z$  (and not on  $x$ ), the motion is periodic with period  $L$ .

A-5. (22, 7, 0, 1, 0, 0, 0, 0, 5, 4, 48, 114)

We are given that

$$r = 1 - \frac{a}{b} - \frac{c}{d} = \frac{bd - ad - bc}{bd} > 0.$$

Thus  $bd - ad - bc$  is a positive integer and so  $r \geq 1/bd$ . We may assume without loss of generality that  $b \leq d$ . If  $b \leq d \leq 1983$ ,  $r \geq 1983^{-2} > 1983^{-3}$ . Since  $a + c \leq 1982$ , if  $1983 \leq b \leq d$ , one has

$$r \geq 1 - \frac{a}{1983} - \frac{c}{1983} \geq 1 - \frac{1982}{1983} = \frac{1}{1983} > \frac{1}{1983^3}.$$

The remaining case is that with  $b < 1983 < d$ . Then the  $d$  that minimizes  $r$  for fixed  $a, b, c$  is  $1 + [bc/(b - a)]$ , where  $[x]$  is the greatest integer in  $x$ . This  $d$  is at most  $1983b$  since  $b - a \geq 1$  and  $c < 1982$  and thus

$$r \geq \frac{1}{bd} \geq \frac{1}{1983b^2} > \frac{1}{1983^3}.$$

Hence we have the desired inequality in all cases.

A-6. (0, 0, 0, 0, 4, 1, 2, 0, 5, 4, 50, 135)

We disprove the assertion. Let  $y_n = 1/(n + 1)\ln(n + 1)$ . Then  $\sum_{n=1}^{\infty} (-1)^{n+1} y_n$  converges to some  $g > 0$  since  $y_n \rightarrow 0$  as  $n \rightarrow \infty$  and  $y_1 > y_2 > \dots$ . Let  $x_n = (-1)^{n+1} y_n/g$ . Then conditions (i) and (iii) are satisfied. Let  $a_0, a_1, \dots$  be positive integers to be made more definite later. Let  $b_0 = 0$  and  $b_{i+1} = b_i + 4a_i$  for  $i = 0, 1, \dots$ . The bijection  $\sigma$  is defined as follows:

$$\begin{aligned} \sigma(n) &= 2n - 1 - b_i \quad \text{for } b_i < n \leq b_i + 2a_i, \\ \sigma(n) &= 2n - b_{i+1} \quad \text{for } b_i + 2a_i < n \leq b_{i+1}. \end{aligned}$$

Then  $0 < \sigma(n) < 2n$  and hence  $|\sigma(n) - n| < n$ . Thus

$$|\sigma(n) - n| \cdot |x_n| < \frac{n}{g(n+1)\ln(n+1)}$$

which implies condition (ii). Let  $C(n) = \sum_{i=1}^n x_i$  and  $D(n) = \sum_{i=1}^n x_{\sigma(i)}$ . Then

$$(A) \quad D(b_i + 2a_i) - C(b_i + 2a_i) = \frac{1}{g} \sum_{j=1}^{a_i} y_{b_i+2j} + \frac{1}{g} \sum_{k=1}^{a_i} y_{b_i+2a_i+2k-1}.$$

Since  $y_2 + y_4 + y_6 + \dots$  diverges to  $+\infty$  by the integral test, the  $a_i$  can be chosen large enough so that the first sum in (A) exceeds 1 for each  $i$ . Then  $D(n) > 1 + C(n)$  for an unbounded sequence of  $n$ 's. Hence  $D(n)$  and  $C(n)$  cannot converge to the same limit.

**B-1.** (56, 46, 23, 0, 0, 0, 0, 18, 5, 5, 48)

The smallest  $n$  is 2. Let  $D$  be the midpoint of side  $AB$ . Cut  $\triangle AMB$  along  $DM$ . Then  $\triangle BMD$  can be placed alongside  $\triangle ADM$ , with side  $BD$  atop side  $AD$ , so as to form a triangle congruent to  $\triangle AMC$ . Since  $\triangle AMB$  need not be congruent to  $\triangle AMC$  in a general  $\triangle ABC$ , there is no method with  $n = 1$ .

**B-2.** (5, 28, 16, 3, 2, 0, 0, 0, 4, 8, 32, 103)

Let  $r = \sqrt{x^2 + y^2}$ ,  $R(m, n) = \{(x, y) : m^2 + n^2 \leq x^2 + y^2\}$ , and

$$I = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A(x, y) e^{-x^2-y^2} dx dy.$$

Let  $\Sigma$  and  $\Sigma'$  denote sums over all integers  $m$  and over all integers  $n$ , respectively. Then

$$\begin{aligned} I &= \Sigma \Sigma' \int \int_{R(m,n)} e^{-x^2-y^2} dx dy \\ &= \Sigma \Sigma' \int_0^{2\pi} \int_{\sqrt{m^2+n^2}}^{\infty} e^{-r^2} r dr d\theta \\ &= \Sigma \Sigma' \int_0^{2\pi} \left[ -\frac{1}{2} e^{-r^2} \right]_{\sqrt{m^2+n^2}}^{\infty} d\theta \\ &= \Sigma \Sigma' \int_0^{2\pi} \frac{1}{2} e^{-m^2-n^2} d\theta \\ &= \Sigma \Sigma' \pi e^{-m^2-n^2} = \pi \left( \Sigma e^{-m^2} \right) \left( \Sigma' e^{-n^2} \right) = \pi(2g - 1)^2. \end{aligned}$$

**B-3.** (82, 18, 5, 0, 0, 0, 0, 3, 8, 6, 40, 39)

Let  $a(n) = [\sqrt{n+1}]$  and  $b(n) = [\sqrt{2n}]$ , where  $[x]$  is the greatest integer in  $x$ . For  $t$  in  $\{1, 2, \dots, a(n)\}$ , there are  $t^2 - 1$  ordered pairs  $(c, d)$  with  $c$  and  $d$  in  $X = \{1, 2, \dots, n\}$  and  $c + d = t^2$ . For  $t$  in  $\{1 + a(n), 2 + a(n), \dots, b(n)\}$ , there are  $2n + 1 - t^2$  ordered pairs  $(c, d)$  with  $c$  and  $d$  in  $X$  and  $c + d = t^2$ . Hence the total number  $F(n)$  of favorable  $(c, d)$  is

$$\begin{aligned} F(n) &= \sum_{t=1}^{a(n)} (t^2 - 1) + \sum_{t=1+a(n)}^{b(n)} (2n + 1 - t^2) \\ &= \left( 2 \sum_{t=1}^{a(n)} t^2 \right) - \left( \sum_{t=1}^{b(n)} t^2 \right) - a(n) + [b(n) - a(n)](2n + 1) \\ &= \frac{2a(n)[1 + a(n)][1 + 2a(n)]}{6} - \frac{b(n)[1 + b(n)][1 + 2b(n)]}{6} \\ &\quad - 2(n + 1)a(n) + (2n + 1)b(n). \end{aligned}$$

Since  $p_n = F(n)/n^2$ ,

$$\begin{aligned} \lim_{n \rightarrow \infty} (p_n \sqrt{n}) &= \lim_{n \rightarrow \infty} F(n)/n^{3/2} \\ &= \frac{2 \cdot 2}{6} \lim_{n \rightarrow \infty} \left( \frac{a(n)}{\sqrt{n}} \right)^3 - \frac{2}{6} \lim_{n \rightarrow \infty} \left( \frac{b(n)}{\sqrt{n}} \right)^3 - 2 \lim_{n \rightarrow \infty} \frac{a(n)}{\sqrt{n}} + 2 \lim_{n \rightarrow \infty} \frac{b(n)}{\sqrt{n}} \\ &= \frac{2}{3} - \frac{1}{3}(\sqrt{2})^3 - 2 + 2\sqrt{2} = \frac{4}{3}(\sqrt{2} - 1). \end{aligned}$$

**B-4.** (2, 7, 5, 1, 13, 3, 3, 1, 0, 2, 36, 128)

Let  $P_k = (n_1 + k)(n_2 + k) \cdots (n_s + k)$ . We are given that  $P_0 | P_k$  for all integers  $k$ .

(a)  $P_0 | P_{-1}$  and  $P_0 | P_1$  together imply  $P_0^2 | (P_{-1}P_1)$  or  $(n_1^2 n_2^2 \cdots n_s^2) | [(n_1^2 - 1)(n_2^2 - 1) \cdots (n_s^2 - 1)]$ .

No  $n_i$  can be zero since  $P_k \neq 0$  for  $k$  sufficiently large. Thus, for each  $i$ ,  $n_i^2 \geq 1$  and  $n_i^2 > n_i^2 - 1 \geq 0$ . Hence  $P_0^2 > P_{-1}P_1 \geq 0$ . This and  $P_0^2 | (P_{-1}P_1)$  imply  $P_{-1}P_1 = 0$ . Then for some  $i$ ,  $|n_i| = 1$ .

(b)  $P_k$  is a polynomial in  $k$  of degree  $s$ . Since  $P_0$  divides each  $P_i$ ,  $P_0$  also divides the  $n$ th difference

$$\sum_{i=0}^s (-1)^i \binom{s}{i} P_i = s!$$

Since  $P_0 > 0$ , this means that  $P_0 \leq s!$ . As  $P_0$  is a product of  $s$  distinct positive integers, it follows that

$$\{n_1, n_2, \dots, n_s\} = \{1, 2, \dots, s\}.$$

**B-5.** (2, 3, 1, 2, 3, 1, 0, 4, 9, 23, 44, 109)

Since the derivative of  $x^{1/x}$  is negative for  $x > e$ ,

(1)  $a^b > b^a$  when  $e \leq a < b$ .

The  $u$ 's are defined so that  $u_0 = e$  and

(2)  $x = (u_0)^{u_1} = (u_1)^{u_2} = (u_2)^{u_3} = \dots$ .

Hence

(3)  $u_{n+1} = (u_n \ln u_{n-1}) / \ln u_n$ .

As  $x > e^e$ ,  $u_1 = \ln x > e = u_0$ . Now  $u_1 > u_0$  implies  $\ln u_1 > \ln u_0$  and then (3) with  $n = 1$  implies  $u_2 < u_1$ . Also, (2) and (1) imply  $(u_1)^{u_2} = (u_0)^{u_1} > (u_1)^{u_0}$ , which gives us  $u_2 > u_0$ . Now  $u_2 < u_1$  and (3) with  $n = 2$  imply  $u_3 > u_2$ . Also (2) and (1) imply  $(u_2)^{u_3} = (u_1)^{u_2} < (u_2)^{u_1}$  and hence  $u_3 < u_1$ . Similarly,  $u_2 < u_4 < u_3$ . Then an easy induction shows that

$$e < u_{2n} < u_{2n+2} < u_{2n+1} < u_{2n-1} \text{ for } n = 1, 2, \dots$$

Thus the monotonic bounded sequences  $u_0, u_2, u_4, \dots$  and  $u_1, u_3, u_5, \dots$  have limits  $a$  and  $b$ , respectively, with  $e < a \leq b$ . Also

$$a^b = \lim_{n \rightarrow \infty} (u_{2n})^{u_{2n+1}} = x = \lim_{n \rightarrow \infty} (u_{2n-1})^{u_{2n}} = b^a.$$

Then  $a^b = b^a$ ,  $e \leq a \leq b$ , and (1) imply  $a = b$ . Hence  $\lim_{n \rightarrow \infty} u_n$  exists and is the unique real number  $g = g(x)$  with  $g > e$  and  $g^g = x$ . Since  $f(y) = y^y$  is continuous and strictly increasing for  $y \geq e$ , its inverse function  $g(x)$  is also continuous.

**B-6.** (10, 3, 7, 0, 0, 0, 1, 8, 10, 29, 133)

Let  $s = (a + b + c)/2$ ,  $t = s - a$ ,  $u = s - b$ ,  $v = s - c$  and similarly for the primed letters.



Using Heron's Formula, the inequality to be proved will follow from

$$(A) \quad \sqrt[4]{stuv} + \sqrt[4]{s't'u'v'} \leq \sqrt[4]{(s+s')(t+t')(u+u')(v+v')}$$

for positive  $s, t, u, v, s', t', u', v'$ . A simpler analogous inequality that might be helpful is

$$(B) \quad \sqrt{xy} + \sqrt{x'y'} \leq \sqrt{(x+x')(y+y')} \quad \text{for } x, y, x', y' \text{ positive.}$$

First we note that (B) follows from the Cauchy Inequality applied to the vectors  $(\sqrt{x}, \sqrt{x'})$  and  $(\sqrt{y}, \sqrt{y'})$  [and also follows from  $(\sqrt{xy'} - \sqrt{x'y})^2 \geq 0$  or from the Inequality on the Means applied to  $xy'$  and  $x'y$ ]. Using (B) with  $x = \sqrt{st}$ ,  $x' = \sqrt{s't'}$ ,  $y = \sqrt{uv}$ ,  $y' = \sqrt{u'v'}$  and reapplying (B) to the new right side, one has

$$\begin{aligned} \sqrt[4]{stuv} + \sqrt[4]{s't'u'v'} &\leq \sqrt{(\sqrt{st} + \sqrt{s't'}) (\sqrt{uv} + \sqrt{u'v'})} \\ &\leq \sqrt{\sqrt{(s+s')(t+t')} \sqrt{(u+u')(v+v')}}. \end{aligned}$$

Since here the rightmost part equals the right side of (A), we have proved (A).

Equality holds in (B) if and only if  $\sqrt{x} : \sqrt{x'} = \sqrt{y} : \sqrt{y'}$  and this holds if and only if  $x : x' = y : y'$ . Hence equality occurs in (A) if and only if  $s : t : u : v = s' : t' : u' : v'$ . It follows that equality occurs in the original inequality if and only if  $a, b, c$  are proportional to  $a', b', c'$ .

## PROGRESS REPORTS

EDITED BY THOMAS BANCHOFF AND RICHARD MILLMAN

It is easy to be too busy to pay attention to what anyone else is doing, but not good. All of us should know, and want to know, what has been discovered since our formal education ended, but new words, and relations between them, are growing too fast to keep up. It is possible for a person to learn of the title of a recent work and of the key words used in it and still not have the faintest idea of what the subject is.

*Progress Reports* is to be an almost periodic column intended to increase everyone's mathematical information about what others have been up to. Each column will report one step forward in the mathematics of our time. The purpose is to inform, more than to instruct: what is the name of the subject, what are some of the words it uses, what is a typical question, what is the answer, who found it. The emphasis will be on concrete questions and answers (theorems), and not on general contexts and techniques (theories). References will be kept minimal: usually they will include only one of the earliest papers in which the answer appears and a more recent exposition of the discovery, whenever one is easily available.

Everyone is invited to nominate subjects to be reported on and authors to prepare the reports. The ground rules are that the principal theorem should be old enough to have been published in the usual sense of that word (and not just circulated by word of mouth or in preprints); it should be of interest to more than just a few specialists; and it should be new enough to have an effect on the mathematical life of the present and near future. In practice most reports will probably be on progress achieved somewhere between 5 and 15 years ago.

## MANIFOLDS WITH THE SAME SPECTRUM

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Many extremely important results in differential geometry concern the effect of the geometry of an object upon its topology. In addition, over the past forty years there has been a great deal of interest in the relationship between analytic quantities and the geometry and topology of