The William Lowell Putnam Mathematical Competition

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we have

$$C = D_{ij} + (2n' - n + (m' - m)(k^2+1))(k^2,k) + (n - n' + m)(0, k^3 + k).$$

Thus, $C \in L_k$ if and only if $D_{ij} \in L_k$. Busy computing yields

$$D_{11} = P_1, \quad D_{12} = P_3, \quad D_{13} = P_3 - (k^2,k), \quad D_{14} = P_1 + (k^2,k),$$
$$D_{21} = P_4, \quad D_{22} = P_2, \quad D_{23} = P_2 - (k^2,k), \quad D_{24} = P_4 + (k^2,k),$$
$$D_{31} = P_1 + (k^2,k), \quad D_{32} = P_3 + (k^2,k), \quad D_{33} = P_3, \quad D_{34} = P_1 + 2(k^2,k),$$
$$D_{41} = P_4 - (k^2,k), \quad D_{42} = P_2 - (k^2,k), \quad D_{43} = P_2 - 2(k^2,k), \quad D_{44} = P_4,$$

showing that $D_{i,j} \in L_k$ for all i and j.

It remains to show that L_2 is the $(2,90°)$-closure H_2 of the points $P_1 = (0,0)$ and $P_2 = (1,0)$. By the first part of the proof it suffices to show that H_2 contains L_2. Of course, $P_3, P_4 \in H_2$. Since L_2 is the union of the images of $S = \{P_1, \dots, P_4\}$ under all translations defined by the vectors

$$\{n(4,2) + m(0,10);\ n,m \in \mathbb{Z}\},$$

it suffices to show that $S - (4,2) \subset H_2$ and $S + (0,10) \subset H_2$. This entertaining construction is left to the reader.

The reader is invited also to devise and investigate similar geometric closure properties of plane (and higher-dimensional) sets.

THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

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The following results of the forty-fourth William Lowell Putnam Mathematical Competition, held on December 3, 1983, have been determined in accordance with the governing regulations. This annual contest is supported by the William Lowell Putnam Prize Fund for the Promotion of Scholarship, left by Mrs. Putnam in memory of her husband, and is held under the auspices of the Mathematical Association of America.

The first prize, five thousand dollars, was awarded to the Department of Mathematics of the **California Institute of Technology**, Pasadena, California. The members of its winning team were: Bradley W. Brock, Charles J. Cuny, and Alan G. Murray; each was awarded a prize of two hundred fifty dollars.

The second prize, two thousand five hundred dollars, was awarded to the Department of Mathematics of **Washington University**, St. Louis, Missouri. The members of its team were: Paul H. Burchard, Edward A. Shpiz, and Richard A. Stong; each was awarded a prize of two hundred dollars.

The third prize, one thousand five hundred dollars, was awarded to the Department of Mathematics of the **University of Waterloo**, Waterloo, Ontario, Canada. The members of its team were: David W. Ash, W. Ross Brown, and Bev I. Cope; each was awarded a prize of one hundred fifty dollars.

The fourth prize, one thousand dollars, was awarded to the Department of Mathematics of **Princeton University**, Princeton, New Jersey. The members of its team were: Gregg N. Patruno, Daniel J. Scales, and Kevin M. Walker; each was awarded a prize of one hundred dollars.

The fifth prize, five hundred dollars, was awarded to the Department of Mathematics of the **University of Chicago**, Chicago, Illinois. The members of its team were Keith A. Ramsay, Michael P. Spertus, and David S. Yuen; each was awarded a prize of fifty dollars.

The five highest-ranking individual contestants, in alphabetical order, were **David W. Ash**, University of Waterloo; **Eric D. Carlson**, Michigan State University; **Noam D. Elkies**, Columbia University; **Michael J. Larsen**, Harvard University; and **Gregg N. Patruno**, Princeton University. Each of these students was designated a Putnam Fellow by the Mathematical Association of America and awarded a prize of five hundred dollars by the Putnam Prize Fund.

The next five highest-ranking individuals, in alphabetical order, were *Thomas O. Andrews*, Yale University; *Joel Friedman*, Harvard University; *Alan G. Murray*, California Institute of Technology; *Richard A. Stong*, Washington University, St. Louis; and *David S. Yuen*, University of Chicago. Each of these students was awarded a prize of two hundred fifty dollars.

The following teams, named in alphabetical order received honorable mention: *University of Alberta*, with team members Arthur B. Baragar, Robert P. Morewood, and David S. Salopek; *Harvard University*, with team members Zachary M. Franco, Joel Friedman, and Michael J. Larsen; *Memorial University of Newfoundland*, with team members Quoc T. Pham, Michael J. Sandys-Wunsch, and Arthur P. Smith; *Queen's University*, with team members Neale Ginsburg, Teddy Hsu, and Michael J. Swain; and *Yale University*, with team members Thomas O. Andrews, Alan S. Edelman, and Nathaniel E. Glasser.

Honorable mention was achieved by the following thirty-four individuals, named in alphabetical order: *Bruce W. K. Brandt*, Harvard University; *Bradley W. Brock*, California Institute of Technology; *W. Ross Brown*, University of Waterloo; *Paul H. Burchard*, Washington University, St. Louis; *Pang-Chieh Chen*, California Institute of Technology; *John J. Chew*, University of Toronto; *Charles J. Cuny*, California Institute of Technology; *David B. Delaney*, Case Western Reserve University; *Stephen A. DiPippo*, Brown University; *Yong Yao Du*, University of Waterloo; *Benji N. Fisher*, Harvard University; *Daniel J. Goldstein*, University of Chicago; *Frederic M. Gourdeau*, Université Laval; *Everett W. Howe*, California Institute of Technology; *Paul S. Hsieh*, Massachusetts Institute of Technology; *Teddy Hsu*, Queen's University; *Jung C. Im*, California Institute of Technology; *Russell G. Impagliazzo*, Wesleyan University; *Eric H. Kawamoto*, California Institute of Technology; *Richard W. Kenyon*, Rice University; *Gary R. Lawlor*, Brigham Young University; *Stephen T. Mark*, Yale University; *David I. McIntosh*, University of Waterloo; *Robert P. Morewood*, University of Alberta; *Howard M. Pollack*, Harvard University; *Keith A. Ramsay*, University of Chicago; *James R. Roche*, University of Notre Dame; *James R. Russell*, Massachusetts Institute of Technology; *Daniel J. Scales*, Princeton University; *Arthur P. Smith*, Memorial University of Newfoundland; *Christopher R. Stover*, Swarthmore College; *John M. Sullivan*, Harvard University; *James C. Yeh*, Princeton University; and *Thomas M. Zavist*, Rice University.

The other individuals who achieved ranks among the top 101, in alphabetical order of their schools, were: University of Alberta, *Arthur B. Baragar*; University of British Columbia, *Lawrence D. Hammick*, *Thomas R. Stevenson*; California Institute of Technology, *Christian G. Bower*, *Jonathan S. Shapiro*; University of California, Davis, *Michael P. Quinn*; University of California, San Diego, *Peter M. De Marzo*; University of California, Santa Barbara, *Emerson S. Fang*, *John R. Kelly*; Case Western Reserve University, *Magnus R. Karlsson*, *Kevin E. Kelso*; University of Chicago, *Geoffrey R. Harris*; Colorado State University, *Jorg A. Brown*; The

Cooper Union, *Tsz Mei Ko*; Harvard University, *Frederick R. Adler*; *Glen D. Ellison*, *Alfred D. Shapere*, *David Wolland*, *David T. Wu*; Harvey Mudd College, *Arthur A. Middleton*; University of Illinois, Urbana-Champaign, *Eric K. Lossin*; Université Laval, *David Bernier*; University of Maryland, Baltimore County, *Gary S. Katzenberger*; Massachusetts Institute of Technology, *Jonathan W. Aronson*, *Andrew E. Gelman*, *Chun-Nip Lee*, *Warren D. Smith*; Memorial University of Newfoundland, *Michael J. Sandys-Wunsch*; Michigan State University, *Erin J. Schram*; University of Michigan, Ann Arbor, *Fred I. Diamond*; University of New Brunswick, *Christian Friesen*; University of North Carolina, Chapel Hill, *Leick D. Robinson*; Northwestern University, *Wayne W. Wheeler*; Oberlin College, *Mark R. Hanisch*, *Iwan Pranata*; University of Pennsylvania, *Mark E. Banilower*, *William A. Graham*; Princeton University, *Troy W. Barbee III*, *Rama R. Kocherlakota*, *Burt J. Totaro*, *Stephen A. Vavasis*, *Kevin M. Walker*; Queen's College of the City University of New York, *Boris Aronov*; Rice University, *Garrett T. Biehle*; Rose-Hulman Institute of Technology, *Daniel W. Johnson*; Stanford University, *Washington Taylor*; University of Texas, Austin, *Andrew Chin*; University of Utah, *Eric M. Weeks*; Washington University, St. Louis, *William H. Paulsen*, *Edward A. Shpiz*; University of Waterloo, *Todd A. Cardno*, *Bev I. Cope*, *Charles S. A. Timar*; University of Wisconsin, Madison, *Chris S. Jantzen*, *John H. Rickert*; University of Wisconsin, Oshkosh, *Douglas G. Kilday*; and Yale University, *Alan S. Edelman*.

There were 2055 individual contestants from 345 colleges and universities in Canada and the United States in the competition of December 3, 1983. Teams were entered by 256 institutions.

The Questions Committee for the forty-fourth competition consisted of Douglas A. Hensley (Chairman), Melvin Hochster, and Bruce Reznick; they composed the problems listed below and were most prominent among those suggesting solutions.

PROBLEMS

Problem A-1

How many positive integers n are there such that n is an exact divisor of at least one of the numbers

$$10^{40}, 20^{30}?$$

Problem A-2

The hands of an accurate clock have lengths 3 and 4. Find the distance between the tips of the hands when that distance is increasing most rapidly.

Problem A-3

Let p be in the set $\{3, 5, 7, 11, \dots\}$ of odd primes and let

$$F(n) = 1 + 2n + 3n^2 + \cdots + (p-1)n^{p-2}.$$

Prove that if a and b are distinct integers in $\{0, 1, 2, \dots, p-1\}$ then $F(a)$ and $F(b)$ are not congruent modulo p, that is, $F(a) - F(b)$ is not exactly divisible by p.

Problem A-4

Let k be a positive integer and let $m = 6k - 1$. Let

$$S(m) = \sum_{j=1}^{2k-1} (-1)^{j+1} \binom{m}{3j-1}.$$

For example with $k = 3$,

$$S(17) = \binom{17}{2} - \binom{17}{5} + \binom{17}{8} - \binom{17}{11} + \binom{17}{14}.$$

Prove that $S(m)$ is never zero. $\left[\text{As usual, } \binom{m}{r} = \frac{m!}{r!(m-r)!}.\right]$

Problem A-5

Prove or disprove that there exists a positive real number u such that $[u^n] - n$ is an even integer for all positive integers n.

Here $[x]$ denotes the greatest integer less than or equal to x.

Problem A-6

Let $\exp(t)$ denote e^t and

$$F(x) = \frac{x^4}{\exp(x^3)} \int_0^x \int_0^{x-u} \exp(u^3 + v^3)\, dv\, du.$$

Find $\lim_{x \to \infty} F(x)$ or prove that it does not exist.

Problem B-1

Let v be a vertex (corner) of a cube C with edges of length 4. Let S be the largest sphere that can be inscribed in C. Let R be the region consisting of all points p between S and C such that p is closer to v than to any other vertex of the cube. Find the volume of R.

Problem B-2

For positive integers n, let $C(n)$ be the number of representations of n as a sum of nonincreasing powers of 2, where no power can be used more than three times. For example, $C(8) = 5$ since the representations for 8 are:

$$8,\quad 4+4,\quad 4+2+2,\quad 4+2+1+1,\quad \text{and}\quad 2+2+2+1+1.$$

Prove or disprove that there is a polynomial $P(x)$ such that $C(n) = [P(n)]$ for all positive integers n; here $[u]$ denotes the greatest integer less than or equal to u.

Problem B-3

Assume that the differential equation

$$y''' + p(x)y'' + q(x)y' + r(x)y = 0$$

has solutions $y_1(x)$, $y_2(x)$, and $y_3(x)$ on the whole real line such that

$$y_1^2(x) + y_2^2(x) + y_3^2(x) = 1$$

for all real x. Let

$$f(x) = (y_1'(x))^2 + (y_2'(x))^2 + (y_3'(x))^2.$$

Find constants A and B such that $f(x)$ is a solution to the differential equation

$$y' + Ap(x)y = Br(x).$$

Problem B-4

Let $f(n) = n + [\sqrt{n}]$ where $[x]$ is the largest integer less than or equal to x. Prove that, for every positive integer

m, the sequence

$$m, f(m), f(f(m)), f(f(f(m))), \ldots$$

contains at least one square of an integer.

Problem B-5

Let $\|u\|$ denote the distance from the real number u to the nearest integer. (For example, $\|2.8\| = .2 = \|3.2\|$.) For positive integers n, let

$$a_n = \frac{1}{n}\int_1^n \left\|\frac{n}{x}\right\| dx.$$

Determine $\lim_{n\to\infty} a_n$. You may assume the identity

$$\tfrac{2}{1}\cdot\tfrac{2}{3}\cdot\tfrac{4}{3}\cdot\tfrac{4}{5}\cdot\tfrac{6}{5}\cdot\tfrac{6}{7}\cdot\tfrac{8}{7}\cdot\tfrac{8}{9}\cdot\cdots = \frac{\pi}{2}.$$

Problem B-6

Let k be a positive integer, let $m = 2^k + 1$, and let $r \neq 1$ be a complex root of $z^m - 1 = 0$. Prove that there exist polynomials $P(z)$ and $Q(z)$ with integer coefficients such that

$$(P(r))^2 + (Q(r))^2 = -1.$$

SOLUTIONS

In the 12-tuples $(n_{10}, n_9, \ldots, n_0, n_{-1})$ following each problem number below, n_i for $10 \geqslant i \geqslant 0$ is the number of students among the top 195 contestants achieving i points for the problem and n_{-1} is the number of those not submitting solutions.

A-1. (155, 26, 3, 0, 0, 0, 0, 0, 4, 5, 1, 1)

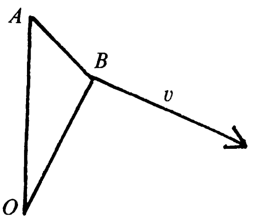
For d and m in $Z^+ = \{1, 2, 3, \ldots\}$, let $d|m$ denote that d is an integral divisor of m. For m in Z^+, let $\tau(m)$ be the number of d in Z^+ such that $d|m$. The number of n in Z^+ such that $n|a$ or $n|b$ is

$$\tau(a) + \tau(b) - \tau(\gcd(a, b)).$$

Also $\tau(p^s q^t) = (s+1)(t+1)$ for p, q, s, t in Z^+ with p and q distinct primes. Thus the desired count is

$$\tau(2^{40}\cdot 5^{40}) + \tau(2^{60}\cdot 5^{30}) - \tau(2^{40}\cdot 5^{30}) = 41^2 + 61\cdot 31 - 41\cdot 31$$
$$= 1681 + 620 = 2301.$$

A-2. (40, 21, 55, 8, 3, 8, 1, 1, 17, 16, 22, 3)



Let OA be the long hand and OB be the short hand. We can think of OA as fixed and OB as rotating at constant speed. Let v be the vector giving the velocity of point B under this assumption. The rate of change of the distance between A and B is the component of v in the direction of AB. Since v is orthogonal to OB and the magnitude of v is constant, this component is maximal when $\measuredangle OBA$ is a right angle, i.e., when the distance AB is $\sqrt{4^2 - 3^2} = \sqrt{7}$.

Alternatively, let x be the distance AB and $\theta = \measuredangle AOB$. By the Law of Cosines,

$$x^2 = 3^2 + 4^2 - 2 \cdot 3 \cdot 4 \cos\theta = 25 - 24\cos\theta.$$

Since $d\theta/dt$ is constant, we may assume units chosen so that θ is also time t. Now

$$2x\frac{dx}{d\theta} = 24\sin\theta,\ \frac{dx}{d\theta} = \frac{12\sin\theta}{\sqrt{25 - 24\cos\theta}}.$$

Since $dx/d\theta$ is an odd function of θ, $|dx/ds|$ is a maximum when $dx/d\theta$ is a maximum or a minimum. Since dx/ds is a periodic differentiable function of θ, $d^2x/ds^2 = 0$ at the extremes for dx/ds. For such θ,

$$12\cos\theta = x\frac{d^2x}{d\theta^2} + \left(\frac{dx}{d\theta}\right)^2 = \left(\frac{dx}{d\theta}\right)^2 = \frac{144\sin^2\theta}{x^2}.$$

Then

$$x^2 = \frac{12\sin^2\theta}{\cos\theta} = \frac{12 - 12\cos^2\theta}{\cos\theta} = 25 - 24\cos\theta,$$

and it follows that

$$12\cos^2\theta - 25\cos\theta + 12 = 0.$$

The only allowable solution for $\cos\theta$ is $\cos\theta = 3/4$ and hence $x = \sqrt{25 - 24\cos\theta} = \sqrt{25 - 18} = \sqrt{7}$.

A-3. (72, 15, 7, 1, 0, 0, 0, 0, 3, 6, 14, 77)

$$F(n) = 1 + 2n + 3n^2 + \cdots + (p-1)n^{p-2},$$
$$nF(n) = n + 2n^2 + \cdots + (p-2)n^{p-2} + (p-1)n^{p-1}.$$

Hence $(1 - n)F(n) = (1 + n + n^2 + \cdots + n^{p-2}) - (p-1)n^{p-1}$ and similarly

$$(1-n)^2F(n) = 1 - n^{p-1} - (1-n)(p-1)n^{p-1} = 1 - p \cdot n^{p-1} + (p-1)n^p.$$

Modulo p, $n^p \equiv n$ by the Little Fermat Theorem and so $(1-n)^2F(n) \equiv 1 - n$. If neither a nor b is congruent to 1 (mod p), $1 - a \not\equiv 1 - b$ and there are distinct reciprocals $(1-a)^{-1}$ and $(1-b)^{-1}$(mod p); then

$$f(a) \equiv (1-a)^{-1}, f(b) \equiv (1-b)^{-1}, f(a) \not\equiv f(b) (\text{mod } p).$$

If one of a and b, say a, is congruent to 1, then $b \not\equiv 0$ (mod p) and so $f(b) \equiv (1-b)^{-1} \not\equiv 0$ (mod p) while

$$f(a) = 1 + 2 + \cdots + (p-1) = p(p-1)/2 \equiv 0 (\text{mod } p).$$

A-4. (15, 2, 7, 3, 0, 1, 0, 3, 4, 5, 29, 126)

Let $\binom{m}{r} = 0$ for $r > m$ and for $r < 0$. For $i = 0, 1, 2$ let

$$T_i(m) = \binom{m}{i} - \binom{m}{i+3} + \binom{m}{i+6} - \binom{m}{i+9} + \cdots.$$

We note that $S(m) = T_2(m) + 1$. Since $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$,

$$T_2(m) = T_2(m-1) + T_1(m-1),\ T_1(m) = T_1(m-1) + T_0(m-1),$$
$$T_0(m) = T_0(m-1) - T_2(m-1).$$

Let the backwards difference operator ∇ be given by $\nabla f(n) = f(n) - f(n-1)$. Then

$$\nabla T_2(m) = T_1(m-1),\ \nabla T_1(m) = T_0(m-1),\ \nabla T_0(m) = -T_2(m-1).$$

These imply that

$$\nabla^3 T_2(m) = \nabla^2 T_1(m-1) = \nabla T_0(m-2) = -T_2(m-3) \text{ for } m \geqslant 3.$$

Expanding $\nabla^3 T_2(m)$, this gives us

(R) $$T_2(m) = 3[T_2(m-1) - T_2(m-2)] \text{ for } m \geqslant 3.$$

When $m = 6k - 1$ with $k \geqslant 1$, we have $m \geqslant 5$. It then follows from (R) that $T_2(m) \equiv 0 \pmod 3$ and hence $S(m) \equiv 1 \pmod 3$. Thus $S(m) \neq 0$.

A-5. (1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 43, 149)

Inductively we define a sequence of integers $3 = a_1, a_2, a_3, \ldots$ and associated intervals $I_n = [(a_n)^{1/n}, (1 + a_n)^{1/n})$ such that $a_n \geqslant 3^n$, $a_n \equiv n \pmod 2$, the sequence $\{(a_n)^{1/n}\}$ is nondecreasing, and $I_n \supseteq I_{n+1}$. When this has been done, $\{(a_n)^{1/n}\}$, being nondecreasing and bounded, will have a limit u which is in I_n for all n. Then $(a_n)^{1/n} \leqslant u < (1 + a_n)^{1/n}$ will imply that $a_n \leqslant u^n < 1 + a_n$ and so $[u^n] = a_n \equiv n \pmod 2$ for all n.

Let $a_1 = 3$. Then $I_1 = [3, 4)$. Let us assume that we have $a_1, a_2, \ldots, a_k$ and $I_1, I_2, \ldots, I_k$ with the desired properties. Let

$$J_k = \left[(a_k)^{(k+1)/k}, (1 + a_k)^{(k+1)/k}\right).$$

Then x is in I_k if and only if x^{k+1} is in J_k. The length of J_k is

$$(1 + a_k)^{(k+1)/k} - (a_k)^{(k+1)/k} \geqslant (1 + a_k - a_k)(a_k)^{1/k} = a_k^{1/k} \geqslant (3^k)^{1/k} = 3.$$

Since the length of J_k is at least 3, J_k contains an interval $L_k = [a_{k+1}, 1 + a_{k+1})$ for some integer a_{k+1} which is congruent to $k + 1 \pmod 2$. Let

$$I_{k+1} = \left[(a_{k+1})^{1/(k+1)}, (1 + a_{k+1})^{1/(k+1)}\right).$$

Since $x \in I_k$ if and only if $x^{k+1} \in J_k$, $x \in I_{k+1}$ if and only if $x^{k+1} \in L_k$, and $J_k \supseteq L_k$, one sees that $I_k \supseteq I_{k+1}$. Also

$$a_{k+1} \geqslant (a_k)^{(k+1)/k} = \left[(a_k)^{1/k}\right]^{k+1} \geqslant 3^{k+1}.$$

This completes the inductive step and shows that the desired u exists.

A-6. (0, 1, 1, 0, 0, 0, 0, 2, 5, 3, 38, 145)

Under the change of variables $s = u - v$ and $t = u + v$, with the Jacobian $\partial(u, v)/\partial(s, t) = 1/2$, $F(x)$ becomes $I(x)/E(x)$ where

$$I(x) = \int_0^x \int_{-t}^{t} \exp\left\{\left(\frac{t+s}{2}\right)^3 + \left(\frac{t-s}{2}\right)^3\right\} ds\, dt$$
$$= \int_0^x \int_{-t}^{t} \exp\left(\frac{1}{4}t^3 + \frac{3}{4}ts^2\right) ds\, dt$$

and $E(x) = 2x^{-4}\exp(x^3)$. Since $I(x)$ and $E(x)$ go to $+\infty$ as x goes to $+\infty$, one can use L'Hôpital's Rule and we have $\lim_{x\to\infty} F(x) = \lim_{x\to\infty}(I'/E')$ where

$$I' = \int_{-x}^{x} \exp\left(\frac{1}{4}x^3 + \frac{3}{4}xs^2\right) ds = \exp(x^3/4) \int_{-x}^{x} \exp(3xs^2/4)\, ds$$

and $E' = (6x^{-2} - 8x^{-5})\exp(x^3)$. In the integral for I', make the change of variable $s = w/\sqrt{x}$,

$ds = dw/\sqrt{x}$, to obtain

$$I' = \frac{\exp(x^3/4)}{\sqrt{x}} \int_{-x\sqrt{x}}^{x\sqrt{x}} \exp(3w^2/4)\, dw.$$

Now

$$\lim_{x\to\infty} F(x) = \lim_{x\to\infty} \frac{I'}{E'} = \lim_{x\to\infty} \frac{\int_{-x\sqrt{x}}^{x\sqrt{x}} \exp(3w^2/4)\, dw}{(6x^{-3/2} - 8x^{-9/2})\exp(3x^3/4)}.$$

We can, and do, use L'Hôpital's rule again to obtain

$$\lim_{x\to\infty} F(x) = \lim_{x\to\infty} \frac{2(3/2)x^{1/2}\exp(3x^2/4)}{[(27/2)x^{1/2} + \cdots]\exp(3x^2/4)} = \frac{2}{9}.$$

B-1. (5, 15, 41, 0, 0, 0, 0, 0, 64, 53, 6, 10, 1)

The diameter of S must be 4 and S must be centered at the center of C. The set of points inside C nearer to v than to another vertex w is the part of that half-space, bounded by the perpendicular bisector of the segment vw, containing v which lies within C. The intersection of these sets is a cube C' bounded by the three facial planes of C through v and the three planes which are perpendicular bisectors of the edges of C at v. These last 3 planes are planes of symmetry for C and S. Hence R is one of 8 disjoint congruent regions whose union is the set of points between S and C, excepting those on the 3 planes of symmetry. Therefore

$$8\,\mathrm{vol}(R) = \mathrm{vol}(C) - \mathrm{vol}(S) = 4^3 - \frac{4\pi}{3}\cdot 2^3,$$

$$\therefore\ \mathrm{vol}(R) = 8 - \frac{4\pi}{3}.$$

B-2. (33, 5, 5, 2, 0, 1, 0, 0, 32, 13, 43, 61)

A representation for $2n$ is of the form

$$2n = e_0 + 2e_1 + 4e_2 + \cdots + 2^k e_k,$$

the e_i in $\{0,1,2,3\}$, and with e_0 in $\{0,2\}$. Then $e_1 + 2e_2 + \cdots + 2^{k-1}e_k$ is a representation for n if $e_0 = 0$ and is a representation for $n-1$ if $e_0 = 2$. Since all representations for n and $n-1$ can be obtained this way,

$$C(2n) = C(n) + C(n-1).$$

Similarly, one finds that

$$C(2n+1) = C(n) + C(n-1) = C(2n).$$

Since $C(1) = 1$ and $C(2) = 2$, an easy induction now shows that $C(n) = [1 + n/2]$.

B-3. (76, 5, 4, 2, 0, 0, 0, 3, 0, 3, 5, 97)

To satisfy the equation, each y_i must have at least 3 derivatives. Here Σ will be a sum with i running over 1, 2, 3. We have $\Sigma y_i^2 = 1$ and $\Sigma(y_i')^2 = f$. Differentiating, one has $\Sigma 2y_iy_i' = 0$ and $\Sigma 2y_i'y_i'' = f'$. Differentiating $\Sigma y_iy_i' = 0$ leads to $\Sigma y_iy_i'' + \Sigma(y_i')^2 = 0$ so $\Sigma y_iy_i'' = -f$. Differentiating this gives us $\Sigma y_i'y_i'' + \Sigma y_iy_i''' = -f'$. This and $\Sigma y_i'y_i'' = f'/2$ leads to $\Sigma y_iy_i''' = -3f'/2$. Multiplying each term of

$$y_i''' + py_i'' + qy_i' + ry_i = 0$$

by y_i and summing gives us

$$-3f'/2 - pf + q \cdot 0 + r = 0.$$

Thus $f' + (2/3)pf = (2/3)r$ and so $A = 2/3 = B$.

B-4. (41, 21, 19, 3, 0, 0, 3, 0, 6, 4, 17, 81)

We can let $m = k^2 + j$, where k and j are integers with $0 \leqslant j \leqslant 2k$, since the next square after k^2 is $k^2 + 2k + 1$; let this j be the *excess* for m. We note that $[\sqrt{m}] = k$ and $f(m) = k^2 + j + k$. If the excess j is 0, m is already a square. Let A consist of the m's with excess j satisfying $0 \leqslant j \leqslant k$ and B consist of the m's with $k < j \leqslant 2k$. If m is in B,

$$f(m) = k^2 + j + k = (k+1)^2 + (j - k - 1),$$

with the excess $j - k - 1$ for $f(m)$ satisfying $0 \leqslant j - k - 1 \leqslant k + 1$, and hence $f(m)$ is either a square or is in A. Thus it suffices to deal with the case in which m is in A. Then $[\sqrt{m + k}] = k$ and

$$f^2(m) = f(f(m)) = f(m + k) = m + 2k = (k+1)^2 + (j - 1).$$

Hence $f^2(m)$ is either a square or an integer in A with excess smaller than that of m. Continuing, one sees that $f^r(m)$ is a square for some r with $0 \leqslant r \leqslant 2j$.

B-5. (77, 16, 14, 0, 0, 0, 0, 0, 9, 4, 16, 59)

By definition of a_n and $\|u\|$,

$$\begin{aligned} a_n &= \sum_{k=1}^{n-1} \frac{1}{n}\left[\int_{2n/(2k+1)}^{n/k} \left(\frac{n}{x} - k\right) dx + \int_{n/(k+1)}^{2n/(2k+1)} \left(k + 1 - \frac{n}{x}\right) dx\right] \\ &= \sum_{k=1}^{n-1}\left[\ln\frac{2k+1}{2k} - \frac{1}{2k+1} + \frac{1}{2k+1} - \ln\frac{2k+2}{2k+1}\right] \\ &= \ln\prod_{k=1}^{n-1}\frac{(2k+1)^2}{2k(2k+2)} = \ln\left[\frac{3}{2}\cdot\frac{3}{4}\cdot\frac{5}{4}\cdot\frac{5}{6}\cdots\frac{(2n-1)}{(2n-2)}\cdot\frac{(2n-1)}{2n}\right]. \end{aligned}$$

Since

$$\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots = \frac{\pi}{2}$$

and $\ln x$ is continuous for $x > 0$, $\lim_{n\to\infty} a_n = \ln(4/\pi)$.

B-6. (3, 0, 0, 1, 0, 0, 0, 1, 1, 20, 16, 153)

Since $r \neq 1$ and $r^m - 1 = (r - 1)(r^{m-1} + r^{m-2} + \cdots + 1) = 0$, one has $r^{m-1} + r^{m-2} + \cdots + 1 = 0$ and so

$$\begin{aligned} -1 &= r(1 + r + r^2 + \cdots + r^{m-2}), \\ -1 &= r(1 + r)(1 + r^2)(1 + r^4)\cdots(1 + r^{(m-1)/2}), \\ -1 &= (r + r^2)(1 + r^2)(1 + r^4)\cdots(1 + r^{(m-1)/2}). \end{aligned}$$

Since $r + r^2 = r^{m+1} + r^2$ with $m + 1 = 2(2^{k-1} + 1)$, each of the factors in the last expression for -1 is a sum of two squares. Their product can be expressed as a sum of two squares by repeated application of the identity

$$(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2.$$

This converts -1 into $P^2 + Q^2$ with each of P and Q a polynomial in r with integer coefficients.