THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

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The following results of the forty-fifth William Lowell Putnam Mathematical Competition, held on December 1, 1984, have been determined in accordance with the governing regulations. This annual contest is supported by the William Lowell Putnam Prize Fund for the Promotion of Scholarship, left by Mrs. Putnam in memory of her husband, and is held under the auspices of the Mathematical Association of America.

The first prize, five thousand dollars, was awarded to both the Department of Mathematics of the **University of California**, Davis, California, and the Department of Mathematics of **Washington University**, St. Louis, Missouri. The teams from these institutions tied. The members of the team from the University of California, Davis, were: John B. Boyland, Robert J. Filippini, and Michael P. Quinn. The members of the team from Washington University, St. Louis, were: William H. Paulsen, Richard A. Stong, Dougin A. Walker. Each member of these teams was awarded a prize of two hundred fifty dollars.

The third prize, one thousand five hundred dollars, was awarded to the Department of Mathematics of **Harvard University**, Cambridge, Massachusetts. The members of its team were: Benji N. Fisher, Howard M. Pollack, and John M. Sullivan; each was awarded a prize of one hundred fifty dollars.

The fourth prize, one thousand dollars, was awarded to the Department of Mathematics of **Princeton University**, Princeton, New Jersey. The members of the team were: Douglas R. Davidson, Gregg N. Patruno, and James C. Yeh; each was awarded a prize of one hundred dollars.

The fifth prize, five hundred dollars, was awarded to the Department of Mathematics of **Yale University**, New Haven, Connecticut. The members of its team were: Thomas O. Andrews, Nathaniel E. Glasser, and Stephen E. Mark; each was awarded a prize of fifty dollars.

The five highest-ranking individual contestants, in alphabetical order, were **Noam D. Elkies**, Columbia University; **Benji N. Fisher**, Harvard University; **Daniel W. Johnson**, Rose-Hulman Institute of Technology; **Michael Reid**, Harvard University; and **Richard A. Stong**, Washington University, St. Louis. Each of these students was designated a Putnam Fellow by the Mathematical Association of America and awarded a prize of five hundred dollars by the Putnam Prize Fund.

The next five highest ranking individuals, in alphabetical order, were *Leland F. Brown*, California Institute of Technology; *Douglas R. Davidson*, Princeton University; *Zachary M. Franco*, Harvard University; *John M. Sullivan*, Harvard University; and *David S. Yuen*, University of Chicago. Each of these students was awarded a prize of two hundred fifty dollars.

The following teams, named in alphabetical order, received honorable mention: *California Institute of Technology*, with team members Everett W. Howe, Jung C. Im, and Eric H. Kawamoto; *University of California*, Berkeley, with team members Michael J. McGrath, Jonathan E. Shapiro, and Christopher S. Welty; *University of Chicago*, with team members Eric K. Lossin, Keith A. Ramsay, and David S. Yuen; *Colorado State University*, with team members Curtis D. Bennett, Jorg A. Brown, and Mark S. Vincent; and *Rose-Hulman Institute of Technology*, with team members Todd G. Fine, Erich J. Friedman, and Daniel W. Johnson.

Honorable mention was achieved by the following thirty-five individuals, named in alphabetical order: *Michael A. Abramson*, Princeton University; *Wayne E. Aitken*, Brigham Young University; *Thomas O. Andrews*, Yale University; *Mark E. Banilower*, University of Pennsylvania; *Arthur B. Baragar*, University of Alberta; *William J. Bruno*, Massachusetts Institute of Technology; *François Destrempes*, Université de Montreal; *Art Duval*, California Institute of Technology; *Duane J. Einfeld*, Dordt College; *Robert J. Filippini*, University of California, Davis; *Leonid Fridman*, Harvard University; *Nathaniel E. Glasser*, Yale University; *Michelangelo Grigni*, Duke University; *George E. Homsy*, University of California, Berkeley; *Liaw Huang*, Rutgers University, New Brunswick; *Douglas S. Jungreis*, Harvard University; *Joe J. Kilian*, Massachusetts Institute of Technology; *Tsz Mei Ko*, The Cooper Union; *Rama R. Kocherlakota*, Princeton University; *Chun-Nip Lee*, Massachusetts Institute of Technology; *Tak Kwan Lee*, University of California, San Diego; *Daniel E. Loeb*, California Institute of Technology; *Michael J. McGrath*, University of California, Berkeley; *Mark E. Meyer*, Indiana University; *Lee A. Newberg*, Massachusetts Institute of Technology; *Jeremy D. Primer*, Princeton University; *Michael P. Quinn*, University of California, Davis; *Keith A. Ramsay*, University of Chicago; *Daniel N. Ropp*, Washington University, St. Louis; *Alistair M. Rucklidge*, University of Toronto; *David S. Salopek*, University of Alberta; *Kenneth W. Shirriff*, University of Waterloo; *Charles S. A. Timar*, University of Waterloo; *Luis G. Valdez-Sanchez*, University of Texas, El Paso; and *Dougin A. Walker*, Washington University, St. Louis.

The other individuals who achieved ranks among the top 102, in alphabetical order of their schools, were: Brigham Young University, *Christopher P. Grant*; California Institute of Technology, *Eric K. Babson*, *William D. Banks*, *Kent J. Cantwell*, *Karen L. Condie*, *William D. Cutrell*, *Jung C. Im*, *James T. Liu*, *Julian West*, *Tad P. White*; University of California, Davis, *John B. Boyland*; University of California, Santa Barbara, *Emerson S. Fang*; Calvin College, *Randy V. Gritter*; Carnegie-Mellon University, *Steven G. DesJardins*, *Jamshid Mahdavi*; University of Chicago, *Geoffrey R. Harris*, *Rainer Hollerbach*, *Thomas R. Lippincott*, *Susan Tolman*; Colorado State University, *Curtis D. Bennett*, *Jorg A. Brown*; Concordia University (G. S. W.), *Chinh Mai*; Cornell University, *Denise E. Freed*; Harvard University, *Glenn D. Ellison*; Haverford College, *Kian-Tat Lim*; University of Illinois, Urbana, *Mark A. Thompson*; University of Kansas, *Glenn G. Chappell*, *Ryan D. Moats*; Université Laval, *Mario M. B. Bergeron*; University of Louisville, *Dung T. Nguyen*; Massachusetts Institute of Technology, *Avrim L. Blum*; Michigan State University, *Frank Sottile*; University of Michigan, Ann Arbor, *Steve Newman*; University of Minnesota, Minneapolis, *Jay A. Jorgenson*; University of Missouri, Rolla, *Ervan E. Darnell*; University of Nebraska, Lincoln, *Bartley E. Goddard*; North Dakota State University, *Jim B. Becker*; University of Notre Dame, *James R. Roche*; Princeton University, *Gregg N. Patruno*, *James C. Yeh*; Rice University, *Garrett T. Biehle*; Rutgers University, New Brunswick, *Scott E. Axelrod*; Simon Fraser University, *Stuart G. Cowan*; Swarthmore College, *John H. Palmieri*; University of Toronto, *Gary F. Baumgartner*, *William J. Rucklidge*; University of Victoria, *Philip H. Spencer*; University of Washington, *Charles N. Curtis*; Washington University, St. Louis, *William H. Paulsen*; University of Waterloo, *Yong Yao Du*, *Alexander T. Kachura*; Wichita State University, *Paul D. Sinclair*; Williams College, *Martin V. Hildebrand*; University of Wisconsin, Eau Claire, *Kenneth J. Dykema*; University of Wisconsin, Milwaukee, *Mark W. Hopkins*; and Yale University, *Ramzi R. Khuri*, *David R. Steinsaltz*.

There were 2149 individual contestants from 350 colleges and universities in Canada and the United States in the competition of December 1, 1984. Teams were entered by 264 institutions.

The Questions Committee for the forty-fifth competition consisted of Melvin Hochster (Chairman), Bruce Reznick, and Richard P. Stanley; they composed the problems listed below and were most prominent among those suggesting solutions.

PROBLEMS

Problem A-1

Let A be a solid $a \times b \times c$ rectangular brick in three dimensions, where $a, b, c > 0$. Let B be the set of all points which are a distance at most one from some point of A (in particular, B contains A). Express the volume of B as a polynomial in a, b, and c.

Problem A-2

Express $\sum_{k=1}^{\infty}(6^k/(3^{k+1}-2^{k+1})(3^k-2^k))$ as a rational number.

Problem A-3

Let n be a positive integer. Let a, b, x be real numbers, with $a \neq b$, and let M_n denote the $2n \times 2n$ matrix whose (i, j) entry m_{ij} is given by

$$m_{ij} = \begin{cases} x & \text{if } i = j, \\ a & \text{if } i \neq j \quad \text{and} \quad i + j \text{ is even}, \\ b & \text{if } i \neq j \quad \text{and} \quad i + j \text{ is odd}. \end{cases}$$

Thus, for example, $M_2 = \begin{pmatrix} x & b & a & b \\ b & x & b & a \\ a & b & x & b \\ b & a & b & x \end{pmatrix}$. Express $\lim_{x \to a} \det M_n/(x-a)^{2n-2}$ as a polynomial in a, b, and n, where $\det M_n$ denotes the determinant of M_n.

Problem A-4

A convex pentagon $P = ABCDE$, with vertices labeled consecutively, is inscribed in a circle of radius 1. Find the maximum area of P subject to the condition that the chords AC and BD be perpendicular.

Problem A-5

Let R be the region consisting of all triples (x, y, z) of nonnegative real numbers satisfying $x + y + z \leqslant 1$. Let $w = 1 - x - y - z$. Express the value of the triple integral

$$\iiint_R x^1 y^9 z^8 w^4 \, dx \, dy \, dz$$

in the form $a!b!c!d!/n!$, where a, b, c, d, and n are positive integers.

Problem A-6. Let n be a positive integer, and let $f(n)$ denote the last nonzero digit in the decimal expansion of $n!$. For instance, $f(5) = 2$.

(a) Show that if $a_1, a_2, \ldots, a_k$ are *distinct* nonnegative integers, then $f(5^{a_1} + 5^{a_2} + \cdots + 5^{a_k})$ depends only on the sum $a_1 + a_2 + \cdots + a_k$.

(b) Assuming part (a), we can define

$$g(s) = f(5^{a_1} + 5^{a_2} + \cdots + 5^{a_k}),$$

where $s = a_1 + a_2 + \cdots + a_k$. Find the least positive integer p for which

$$g(s) = g(s + p), \quad \text{for all } s \geqslant 1,$$

or else show that no such p exists.

Problem B-1

Let n be a positive integer, and define

$$f(n) = 1! + 2! + \cdots + n!.$$

Find polynomials $P(x)$ and $Q(x)$ such that

$$f(n+2) = P(n)f(n+1) + Q(n)f(n),$$

for all $n \geqslant 1$.

Problem B-2. Find the minimum value of

$$(u-v)^2 + \left(\sqrt{2-u^2} - \frac{9}{v}\right)^2$$

for $0 < u < \sqrt{2}$ and $v > 0$.

Problem B-3

Prove or disprove the following statement: If F is a finite set with two or more elements, then there exists a binary operation $*$ on F such that for all x, y, z in F,

(i) $x * z = y * z$ implies $x = y$ (right cancellation holds),

and

(ii) $x * (y * z) \neq (x * y) * z$ (*no* case of associativity holds).

Problem B-4

Find, with proof, all real-valued functions $y = g(x)$ defined and *continuous* on $[0, \infty)$, positive on $(0, \infty)$, such that for all $x > 0$ the y-coordinate of the centroid of the region

$$R_x = \{(s,t) | 0 \leqslant s \leqslant x, \quad 0 \leqslant t \leqslant g(s)\}$$

is the same as the average value of g on $[0, x]$.

Problem B-5

For each nonnegative integer k, let $d(k)$ denote the number of 1's in the binary expansion of k (for example, $d(0) = 0$ and $d(5) = 2$). Let m be a positive integer. Express

$$\sum_{k=0}^{2^m - 1} (-1)^{d(k)} k^m$$

in the form $(-1)^m a^{f(m)}(g(m))!$, where a is an integer and f and g are polynomials.

Problem B-6

A sequence of convex polygons $\{P_n\}$, $n \geqslant 0$, is defined inductively as follows. P_0 is an equilateral triangle with sides of length 1. Once P_n has been determined, its sides are trisected; the vertices of P_{n+1} are the *interior* trisection points of the sides of P_n. Thus, P_{n+1} is obtained by cutting corners off P_n, and P_n has $3 \cdot 2^n$ sides. (P_1 is a regular hexagon with sides of length $1/3$.)

Express $\lim_{n \to \infty} \text{Area}(P_n)$ in the form $\sqrt{a}/b$, where a and b are positive integers.

SOLUTIONS

In the 12-tuples $(n_{10}, n_9, \ldots, n_0, n_{-1})$ following each problem number below, n_i for $10 \geqslant i \geqslant 0$ is the number of students among the top 205 contestants achieving i points for the problem and n_{-1} is the number of those not submitting solutions.

A-1. (146, 3, 33, 0, 0, 0, 0, 0, 21, 1, 1, 0)

The set B can be partitioned into the following sets:

(i) A itself, of volume abc;

(ii) two $a \times b \times 1$ bricks, two $a \times c \times 1$ bricks, and two $b \times c \times 1$ bricks, of total volume $2ab + 2ac + 2bc$;

(iii) four quarter-cylinders of length a and radius 1, four quarter-cylinders of length b and radius 1, and four quarter-cylinders of length c and radius 1, of total volume $(a + b + c)\pi$;

(iv) eight spherical sectors, each consisting of one-eighth of a sphere of radius 1, of total volume $4\pi/3$.

Hence the volume of B is

$$abc + 2(ab + ac + bc) + \pi(a + b + c) + \frac{4\pi}{3}.$$

A-2. (73, 3, 12, 1, 0, 11, 3, 17, 2, 1, 16, 66)

Let $S(n)$ denote the nth partial sum of the given series. Then

$$S(n)=\sum_{k=1}^{n}\left[\frac{3^k}{3^k-2^k}-\frac{3^{k+1}}{3^{k+1}-2^{k+1}}\right]=3-\frac{3^{n+1}}{3^{n+1}-2^{n+1}},$$

and the series converges to $\lim_{n\to\infty}S(n)=2$.

A-3. (38, 1, 7, 0, 0, 0, 0, 1, 7, 0, 55, 96)

Let $N=M_n]_{x=a}$. N has rank 2, so that 0 is an eigenvalue of multiplicity $2n-2$. Let $\mathbf{e}$ denote the $2n\times 1$ column vector of 1's. Notice that $N\mathbf{e}=n(a+b)\mathbf{e}$, and therefore $n(a+b)$ is an eigenvalue. The trace of N is $2na$, and therefore the remaining eigenvalue is $2na-n(a+b)=n(a-b)$. [Note: This corresponds to the eigenvector $\mathbf{f}$, where $f_{i,1}=(-1)^{i+1}$, $i=1,\dots,2n$.]

The preceding analysis implies that the characteristic equation of N is

$$\det(N-\lambda I)=\lambda^{2n-2}(\lambda-n(a+b))(\lambda-n(a-b)).$$

Let $\lambda=a-x$. Then

$$\det M_n=\det(N-(a-x)I)=(a-x)^{2n-2}(a-x-n(a+b))(a-x-n(a-b)).$$

It follows that

$$\lim_{x\to a}\frac{\det M_n}{(x-a)^{2n-2}}=\lim_{x\to a}(a-x-n(a+b))(a-x-n(a-b))=n^2(a^2-b^2).$$

A-4. (7, 3, 4, 3, 0, 0, 0, 6, 3, 4, 82, 93)

Let $\theta=\operatorname{Arc}AB$, $\alpha=\operatorname{Arc}DE$, and $\beta=\operatorname{Arc}EA$. Then $\operatorname{Arc}CD=\pi-\theta$ and $\operatorname{Arc}BC=\pi-\alpha-\beta$.

The area of P, in terms of the five triangles from the center of the circle is

$$\frac{1}{2}\sin\theta+\frac{1}{2}\sin(\pi-\theta)+\frac{1}{2}\sin\alpha+\frac{1}{2}\sin\beta+\frac{1}{2}\sin(\pi-\alpha-\beta).$$

This is maximized when $\theta=\pi/2$ and $\alpha=\beta=\pi/3$. Thus, the maximum area is

$$\frac{1}{2}\cdot 1+\frac{1}{2}\cdot 1+\frac{1}{2}\frac{\sqrt{3}}{2}+\frac{1}{2}\frac{\sqrt{3}}{2}+\frac{1}{2}\frac{\sqrt{3}}{2}=1+\frac{3}{4}\sqrt{3}.$$

A-5. (52, 4, 4, 1, 0, 1, 0, 0, 5, 3, 39, 96)

For $t>0$, let R_t be the region consisting of all triples (x,y,z) of nonnegative real numbers satisfying $x+y+z\leqslant t$. Let

$$I(t)=\iiint_{R_t}x^1y^9z^8(t-x-y-z)^4\,dx\,dy\,dz$$

and make the change of variables $x=tu$, $y=tv$, $z=tw$. We see that $I(t)=I(1)t^{25}$.

Let $J=\int_0^\infty I(t)e^{-t}\,dt$. Then

$$J=\int_0^\infty I(1)t^{25}e^{-t}\,dt=I(1)\Gamma(26)=I(1)25!.$$

It is also the case that

$$J=\int_{t=0}^{\infty}\iiint_{R_t}e^{-t}x^1y^9z^8(t-x-y-z)^4\,dx\,dy\,dz\,dt.$$

Let $s=t-x-y-z$. Then

$$J=\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-s}e^{-x}e^{-y}e^{-z}x^1y^9z^8s^4\,dx\,dy\,dz\,ds=\Gamma(2)\Gamma(10)\Gamma(9)\Gamma(5)=1!9!8!4!.$$

The integral we desire is $I(1)=J/25!=1!9!8!4!/25!$.

A-6. (3, 1, 0, 1, 1, 0, 0, 1, 2, 0, 23, 173)

(a) All congruences are modulo 10.

LEMMA. $f(5n) \equiv 2^n f(n)$.

Proof. We have

$$(*) \qquad (5n)! = 10^n n! \prod_{i=0}^{n-1} \frac{(5i+1)(5i+2)(5i+3)(5i+4)}{2}.$$

If i is even, then

$$\frac{1}{2}(5i+1)(5i+2)(5i+3)(5i+4) \equiv \frac{1}{2}(1 \cdot 2 \cdot 3 \cdot 4) \equiv 2,$$

and if i is odd, then

$$\frac{1}{2}(5i+1)(5i+2)(5i+3)(5i+4) \equiv \frac{1}{2}(6 \cdot 7 \cdot 8 \cdot 9) \equiv 2.$$

Thus the entire product above is congruent to 2^n. From $(*)$ it is clear that the largest power of 10 dividing $(5n)!$ is the same as the largest power of 10 dividing $10^n n!$, and the proof follows.

We now show by induction on $5^{a_1} + \cdots + 5^{a_k}$ that

$$f(5^{a_1} + \cdots + 5^{a_k}) \equiv 2^{a_1 + \cdots + a_k}$$

(which depends only on $a_1 + \cdots + a_k$ as desired).
This is true for $5^{a_1} + \cdots + 5^{a_k} = 1$, since $f(5^0) \equiv 2^0 \equiv 1$.

CASE 1. All $a_i > 0$. By the lemma and induction,

$$\begin{aligned} f(5^{a_1} + \cdots + 5^{a_k}) &\equiv 2^{5^{a_1-1} + \cdots + 5^{a_k-1}} f(5^{a_1-1} + \cdots + 5^{a_k-1}) \\ &\equiv 2^k \cdot 2^{(a_1-1) + \cdots + (a_k-1)} \qquad (\text{since } 2^{5^i} \equiv 2 \text{ for } i \geqslant 0) \\ &\equiv 2^{a_1 + \cdots + a_k}. \end{aligned}$$

CASE 2. Some $a_i = 0$, say $a_1 = 0$. Now

$$(1+5m)! = (1+5m)(5m)!,$$

so $f(1+5m) \equiv (1+5m)f(5m)$. But $f(5m)$ is even for $m \geqslant 1$ since $(5m)!$ is divisible by a higher power of 2 than of 5. But

$$(1+5m) \cdot (2j) \equiv 2j,$$

so $f(1+5m) \equiv f(5m)$. Letting $m = 5^{a_2-1} + \cdots + 5^{a_k-1}$, the proof follows by induction.

(b) The least $p \geqslant 1$ for which $2^{s+p} \equiv 2^2$ for all $s \geqslant 1$ is $p = 4$.

B-1. (179, 9, 6, 0, 0, 0, 0, 1, 0, 0, 4, 6)

We have

$$f(n+2) - f(n+1) = (n+2)! = (n+2)(n+1)! = (n+2)[f(n+1) - f(n)].$$

It follows that we can take $P(x) = x + 3$ and $Q(x) = -x - 2$.

B-2. (62, 13, 19, 6, 0, 0, 0, 0, 15, 15, 34, 41)

The problem asks for the minimum distance between the quarter of the circle $x^2 + y^2 = 2$ in the open first quadrant and the half of the hyperbola $xy = 9$ in that quadrant. Since the tangents to the respective curves at $(1,1)$ and $(3,3)$ separate the curves and are both perpendicular to $x = y$, the minimum distance is 8.

B-3. (93, 16, 1, 1, 0, 0, 0, 1, 2, 5, 33, 53)

The statement is true. Let ϕ be any bijection on F with no fixed points, and set $x * y = \phi(x)$.

B-4. (2, 11, 2, 8, 0, 1, 0, 25, 4, 7, 48, 97)

Such a function must satisfy

$$\frac{\int_0^x \frac{1}{2} g^2(t)\, dt}{\int_0^x g(t)\, dt} = \frac{1}{x} \int_0^x g(t)\, dt,$$

or equivalently,

$$\int_0^x \frac{1}{2} g^2(t)\, dt = \frac{1}{x} \left[\int_0^x g(t)\, dt \right]^2.$$

Let $z(x) = \int_0^x g(t)\, dt$. Then $z'(x) = g(x)$ and we have

$$\int_0^x \frac{1}{2} (z')^2\, dt = \frac{z^2}{x}, \qquad x > 0.$$

Differentiating, we have

$$\frac{1}{2}(z')^2 = \frac{x \cdot 2zz' - z^2}{x^2}, \qquad x > 0,$$

$$x^2(z')^2 - 4xzz' + 2z^2 = 0, \qquad x > 0,$$

$$(xz' - r_1 z)(xz' - r_2 z) = 0, \qquad x > 0,$$

where $r_1 = 2 + \sqrt{2}$ and $r_2 = 2 - \sqrt{2}$.

Now x, z', and z are continuous and $z > 0$, so the last equation implies t
$r = r_1$ or $r = r_2$. Separating variables, we have $z'/z = r/x$ and it follows th

$$\ln z = r \ln x + C_0,$$

or equivalently, $z = C_1 x^r$, $C_1 > 0$. Differentiating, we have $z' = g(x) = Cx^{r-1}$, $C > 0$. But g is continuous on $[0, \infty)$ and therefore we cannot have $r = r_2$ (because $r_2 - 1 = 1 - \sqrt{2} < 0$). Thus

$$g(x) = Cx^{1+\sqrt{2}}, \quad C > 0,$$

and one can check that such $g(x)$ do satisfy all the conditions of the problem.

B-5. (5, 1, 2, 0, 0, 3, 0, 4, 9, 11, 17, 153)

Define

$$D(x) = (1 - x)(1 - x^2)(1 - x^4) \cdots (1 - x^{2^{n-1}}).$$

Since binary expansions are unique, each monomial x^k ($0 \leqslant k \leqslant 2^n - 1$) appears exactly once in the expansion of $D(x)$, with coefficient $(-1)^{d(k)}$. That is,

$$D(x) = \sum_{k=0}^{2^n - 1} (-1)^{d(k)} x^k.$$

Applying the operator $\left(x \frac{d}{dx} \right)$ to $D(x)$ m times, we obtain

$$\left(x \frac{d}{dx} \right)^m D(x) = \sum_{k=0}^{2^n - 1} (-1)^{d(k)} k^m x^k,$$

so that

$$\left(x\frac{d}{dx}\right)^m D(x)\Big]_{x=1} = \sum_{k=0}^{2^n-1}(-1)^{d(k)}k^m.$$

Define $F(x) = D(x+1)$, so that

$$\left(x\frac{d}{dx}\right)^m D(x)\Big]_{x=1} = \left[(x+1)\frac{d}{dx}\right]^m F(x)\Big]_{x=0}.$$

But

$$F(x) = \prod_{\alpha=1}^{m}\left[1-(x+1)^{2^{\alpha-1}}\right] = \prod_{\alpha=1}^{m}\left[-2^{\alpha-1}x + O(x^2)\right], \quad (x\to 0),$$
$$= (-1)^m 2^{m(m-1)/2}x^m + O(x^{m+1}),$$

and by observing that $[(x+1)d/dx]x^n = nx^n + nx^{n-1}$, we see that

$$\left[(x+1)\frac{d}{dx}\right]^m\left(Ax^m + O(x^{m+1})\right) = m!A + O(x).$$

So

$$\left[(x+1)\frac{d}{dx}\right]^m F(x)\Big]_{x=0} = (-1)^m 2^{m(m-1)/2}m! + O(x)\big]_{x=0} = (-1)^m 2^{m(m-1)/2}m!.$$

B-6. (7, 3, 3, 0, 0, 0, 0, 0, 6, 1, 48, 137)

Suppose that $\vec{u}$ and $\vec{v}$ are consecutive edges in P_n. Then $\vec{u}/3$, $(\vec{u}+\vec{v})/3$, and $\vec{v}/3$ are consecutive edges in P_{n+1}. Further,

$$\frac{1}{2}\left\|\frac{\vec{u}}{3}\times\frac{\vec{v}}{3}\right\| = \frac{1}{18}\|\vec{u}\times\vec{v}\|$$

is removed at this corner in making P_{n+1}. But at the next step, the amount from these three consecutive edges is

$$\frac{1}{2}\left\|\frac{\vec{u}}{9}\times\frac{\vec{u}+\vec{v}}{9}\right\| + \frac{1}{2}\left\|\frac{\vec{u}+\vec{v}}{9}\times\frac{\vec{v}}{9}\right\| = \frac{1}{81}\|\vec{u}\times\vec{v}\|.$$

Thus, the amount removed in the $(k+1)$st snip is $2/9$ times the amount removed in the kth.

Note that one-third of the original area is removed at the first step. Thus, the amount removed altogether is

$$\frac{1}{3}\left[1 + (2/9) + (2/9)^2 + \cdots\right] = \frac{1}{3}\cdot\frac{9}{7} = \frac{3}{7}$$

of the original area. Since the original area is $\sqrt{3}/4$, we have

$$\lim_{n\to\infty}\text{Area } P_n = \frac{4}{7}\cdot\frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{7}.$$

The curve in this problem has been studied extensively by Georges de Rham. (See "Un peu de mathématiques à propos d'une courbe plane," *Elem. Math.*, 2 (1947), 73–76, 89–97; "Sur une courbe plane," *J. Math. Pures Appl.*, 35 (1956), 25–42; and "Sur les courbes limites de polygones obtenus par trisection," *Enseign. Math.*, 5 (1959), 29–43.) Among de Rham's results are the following. The limiting curve is C^1 with zero curvature almost everywhere, but every subarc contains points where the curvature is infinite. Consequently, the curve is nowhere analytic. De Rham parametrizes pieces of the curve so that the tangent vector is intimately related to the Minkowski ?-function. If the construction is repeated, but with each edge divided in the ratio (1/4, 1/2, 1/4) rather than (1/3, 1/3, 1/3), then the resulting limit curve *is* analytic, consisting of piecewise parabolic arcs.