THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

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The following results of the forty-sixth William Lowell Putnam Mathematical Competition, held on December 7, 1985, have been determined in accordance with the governing regulations. This annual contest is supported by the William Lowell Putnam Prize Fund for the Promotion of Scholarship, left by Mrs. Putnam in memory of her husband, and is held under the auspices of the Mathematical Association of America.

The first prize, five thousand dollars, was awarded to the Department of Mathematics of **Harvard University**, Cambridge, Massachusetts. The members of the winning team were: Glenn D. Ellison, Douglas S. Jungreis, and Michael Reid; each was awarded a prize of two hundred fifty dollars.

The second prize, two thousand five hundred dollars, was awarded to the Department of Mathematics of **Princeton University**, Princeton, New Jersey. The members of the winning team were: Michael A. Abramson, Douglas R. Davidson, and James C. Yeh; each was awarded a prize of two hundred dollars.

The third prize, one thousand five hundred dollars, was awarded to the Department of Mathematics of the **University of California**, Berkeley, California. The members of the winning team were: Michael J. McGrath, David P. Moulton, and Jonathan E. Shapiro; each was awarded a prize of one hundred fifty dollars.

The fourth prize, one thousand dollars, was awarded to the Department of Mathematics of **Rice University**, Houston, Texas. The members of its team were: Charles R. Ferenbaugh, Thomas M. Hyer, and Thomas M. Zavist; each was awarded a prize of one hundred dollars.

The fifth prize, five hundred dollars, was awarded to the Department of Mathematics of the **University of Waterloo**, Waterloo, Ontario, Canada. The members of its team were: David W. Ash, Yong Yao Du, and Kenneth W. Shirriff; each was awarded a prize of fifty dollars.

The five highest-ranking individual contestants, in alphabetical order, were **Martin V. Hildebrand**, Williams College; **Everett W. Howe**, California Institute of Technology; **Douglas S. Jungreis**, Harvard University; **Bjorn M. Poonen**, Harvard University; and **Keith A. Ramsay**, University of Chicago. Each of these students was designated a Putnam Fellow by the Mathematical Association of America and awarded a prize of five hundred dollars by the Putnam Prize Fund.

The next five highest-ranking individuals, in alphabetical order, were *David W. Ash*, University of Waterloo; *Waldemar P. Horwat*, Massachusetts Institute of Technology; *Greg J. Kuperberg*, Harvard University; *John M. Steinke*, Rice University; and *David I. Zuckerman*, Harvard University. Each of these students was awarded a prize of two hundred fifty dollars.

The following teams, named in alphabetical order, received honorable mention: *California Institute of Technology*, with team members Leland F. Brown, Art Duval, and Daniel E. Loeb; *Rensselaer Institute of Technology*, with team members Brenden A. Del Favero, John D. Moores, and Bart C. Vashaw; *Rutgers University*, New Brunswick, with team members Scott E. Axelrod, Liaw Huang, and William M. Wells; *Washington University*, St. Louis, with team members Anders W. McCarthy, Daniel N. Ropp, and Dougin A. Walker; and *Yale University*, with team members Thomas Andrews, Ramzi R. Khuri, and David R. Steinsaltz.

Honorable mention was achieved by the following thirty-four individuals, named in alphabetical order: *Michael A. Abramson*, Princeton University; *Jorg Anthony Brown*, Colorado State

University; *Glenn G. Chappell*, University of Kansas; *Stanley Chen*, California Institute of Technology; *John J. Chew*, University of Toronto; *Constantine N. Costes*, Harvard University; *Douglas R. Davidson*, Princeton University; *Glenn D. Ellison*, Harvard University; *Adam F. Falk*, University of North Carolina, Chapel Hill; *Christopher P. Grant*, Brigham Young University; *Mark C. Hamburg*, University of Michigan, Ann Arbor; *Liaw Huang*, Rutgers University, New Brunswick; *Thomas M. Hyer*, Rice University; *William Carl Jockusch*, Carleton College; *Daniel W. Johnson*, Rose-Hulman Institute of Technology; *Jamshid Mahdavi*, Carnegie-Mellon University; *Michael J. McGrath*, University of California, Berkeley; *David J. Moews*, Harvard University; *David P. Moulton*, University of California, Berkeley; *Lee A. Newberg*, Massachusetts Institute of Technology; *Ken E. Newman*, Washington University, St. Louis; *Steve Newman*, University of Michigan, Ann Arbor; *Michael Reid*, Harvard University; *Daniel N. Ropp*, Washington University, St. Louis; *Randall G. Rose*, Princeton University; *David B. Secrest*, University of Illinois, Champaign-Urbana; *Robert E. Shapire*, Brown University; *Randall D. Smith*, University of Chicago; *Eric H. Veach*, University of Waterloo; *Minh Tue Vo*, University of Waterloo; *Dougin A. Walker*, Washington University, St. Louis; *Christopher S. Welty*, University of California, Berkeley; *James C. Yeh*, Princeton University; and *Thomas M. Zavist*, Rice University.

The other individuals who achieved ranks among the top 102, in alphabetical order of their schools, were: Bethel College, *Jonathan P. McCammond*; University of British Columbia, *Marvin S. Lee*; California Institute of Technology, *Eric K. Babson*, *William D. Banks*, *Leland F. Brown*, *Earl A. Hubbell*, *Kenneth F. Kelley*, *James T. Liu*, *David J. Nice*, *Theron W. Stanford*, *Steven B. Waltman*; University of California, Berkeley, *Jonathan E. Shapiro*; University of California, Davis, *John B. Boyland*, *Michael P. Quinn*; University of California, Los Angeles, *Joshua R. Zucker*; University of California, Santa Barbara, *Emerson S. Fang*; Carleton University, *Serge Elnitsky*; Case Western Reserve University, *Patrick T. Headley*; University of Chicago, *Robert P. Stingley*; Concordia University, *Chinh Mai*; Cooper Union, *George Dedaj*; Duke University, *Michelangelo Grigni*; Harvard University, *David N. Esch*, *Jonathan L. Feng*, *Howard M. Pollack*; Université Laval, *Mario Bergeron*; Massachusetts Institute of Technology, *Jonathan W. Aronson*, *David T. Blackston*, *Anthony J. Camire*, *James R. Rauen*; University of Minnesota, Minneapolis, *Bruce W. K. Brandt*; University of New Mexico, *William J. Goldman*; State University of New York, Buffalo, *Eric J. Cockayne*; State University of New York, Stony Brook, *Robert A. Hockberg*; Oberlin College, *Gregory S. Ludwig*; Princeton University, *Stephen J. Fromm*; Queen's University, *Neale Ginsburg*; Rensselaer Institute of Technology, *Brendan A. Del Favero*; Rice University, *Charles R. Ferenbaugh*; University of Rochester, *Sze-Him Ng*; Rutgers University, New Brunswick, *Scott E. Axelrod*, *William M. Wells*; Texas A & M University, *Mark G. Yarbrough*; University of Toronto, *Gary F. Baumbartner*, *William J. Rucklidge*; Vanderbilt University, *Byron Lee Walden*; Washington State University, *Dale A. Nichols*; University of Washington, *Micah E. Fogel*; University of Waterloo, *Yong Yao Du*, *Peter A. Heeman*, *Kenneth W. Shirriff*; Yale University, *Tony J. Fisher*, *Ramzi R. Khuri*, *Kamal F. Khuri-Makdisi*, *Moses G. Klein*, *Richard S. Margolin*, *David R. Steinsaltz*; Youngstown State University, *John P. Dalbec*.

There were 2079 individual contestants from 348 colleges and universities in Canada and the United States in the competition of December 7, 1985. Teams were entered by 264 institutions.

The Questions Committee for the forty-sixth competition consisted of Bruce Reznick (Chair), Richard P. Stanley and Harold M. Stark; they composed the problems listed below and were most prominent among those suggesting solutions.

PROBLEMS

Problem A-1

Determine, with proof, the number of ordered triples (A_1, A_2, A_3) of sets which have the property that

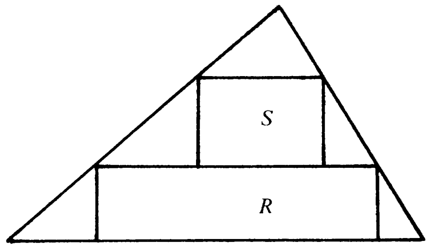
(i) $A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and

(ii) $A_1 \cap A_2 \cap A_3 = \varnothing$,

where $\varnothing$ denotes the empty set. Express the answer in the form $2^a 3^b 5^c 7^d$, where a, b, c and d are nonnegative integers.

Problem A-2

Let T be an acute triangle. Inscribe a pair R, S of rectangles in T as shown:



Let $A(X)$ denote the area of polygon X. Find the maximum value, or show that no maximum exists, of $\dfrac{A(R)+A(S)}{A(T)}$, where T ranges over all triangles and R, S over all rectangles as above.

Problem A-3

Let d be a real number. For each integer $m \geqslant 0$, define a sequence $\{a_m(j)\}$, $j = 0, 1, 2, \ldots$ by the condition

$$a_m(0) = d/2^m, \quad \text{and} \quad a_m(j+1) = (a_m(j))^2 + 2a_m(j), \quad j \geqslant 0.$$

Evaluate $\lim_{n \to \infty} a_n(n)$.

Problem A-4

Define a sequence $\{a_i\}$ by $a_1 = 3$ and $a_{i+1} = 3^{a_i}$ for $i \geqslant 1$. Which integers between 00 and 99 inclusive occur as the last two digits in the decimal expansion of infinitely many a_i?

Problem A-5

Let $I_m = \int_0^{2\pi} \cos(x)\cos(2x) \cdots \cos(mx)\, dx$. For which integers m, $1 \leqslant m \leqslant 10$, is $I_m \neq 0$?

Problem A-6

If $p(x) = a_0 + a_1 x + \cdots + a_m x^m$ is a polynomial with real coefficients a_i, then set

$$\Gamma(p(x)) = a_0^2 + a_1^2 + \cdots + a_m^2.$$

Let $f(x) = 3x^2 + 7x + 2$. Find, with proof, a polynomial $g(x)$ with real coefficients such that

(i) $g(0) = 1$,

and

(ii) $\Gamma(f(x)^n) = \Gamma(g(x)^n)$,

for every integer $n \geqslant 1$.

Problem B-1

Let k be the smallest positive integer with the following property:

There are distinct integers m_1, m_2, m_3, m_4, m_5 such that the polynomial

$$p(x) = (x - m_1)(x - m_2)(x - m_3)(x - m_4)(x - m_5)$$

has exactly k nonzero coefficients.

Find, with proof, a set of integers m_1, m_2, m_3, m_4, m_5 for which this minimum k is achieved.

Problem B-2

Define polynomials $f_n(x)$ for $n \geqslant 0$ by $f_0(x) = 1$, $f_n(0) = 0$ for $n \geqslant 1$, and

$$\frac{d}{dx}(f_{n+1}(x)) = (n+1)f_n(x+1)$$

for $n \geqslant 0$. Find, with proof, the explicit factorization of $f_{100}(1)$ into powers of distinct primes.

Problem B-3

Let

$$\begin{array}{cccc} a_{1,1} & a_{1,2} & a_{1,3} & \cdots \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

be a doubly infinite array of positive integers, and suppose each positive integer appears exactly eight times in the array. Prove that $a_{m,n} > mn$ for some pair of positive integers (m, n).

Problem B-4

Let C be the unit circle $x^2 + y^2 = 1$. A point p is chosen randomly on the circumference C and another point q is chosen randomly from the interior of C (these points are chosen independently and uniformly over their domains). Let R be the rectangle with sides parallel to the x-and y-axes with diagonal pq. What is the probability that no point of R lies outside of C?

Problem B-5

Evaluate $\int_0^\infty t^{-1/2}e^{-1985(t+t^{-1})}\,dt$. You may assume that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$.

Problem B-6

Let G be a finite set of real $n \times n$ matrices $\{M_i\}$, $1 \leqslant i \leqslant r$, which form a group under matrix multiplication. Suppose that $\Sigma_{i=1}^r \mathrm{tr}(M_i) = 0$, where $\mathrm{tr}(A)$ denotes the trace of the matrix A. Prove that $\Sigma_{i=1}^r M_i$ is the $n \times n$ zero matrix.

SOLUTIONS

In the 12-tuples $(n_{10}, n_9, \ldots, n_0, n_{-1})$ following each problem number below, n_i for $10 \geqslant i \geqslant 0$ is the number of students among the top 201 contestants achieving i points for the problem and n_{-1} is the number of those not submitting solutions.

A-1. (125, 6, 0, 0, 0, 0, 0, 0, 0, 0, 61, 9)

Every integer, $1 \leqslant i \leqslant 10$, falls into one of six mutually disjoint classes: $A_1 \cap \bar{A}_2 \cap \bar{A}_3$, $\bar{A}_1 \cap A_2 \cap \bar{A}_3$, $\bar{A}_1 \cap \bar{A}_2 \cap A_3$, $\bar{A}_1 \cap A_2 \cap A_3$, $A_1 \cap \bar{A}_2 \cap A_3$, and $A_1 \cap A_2 \cap \bar{A}_3$; hence there are $6^{10} = 2^{10}3^{10}$ different ordered triples.

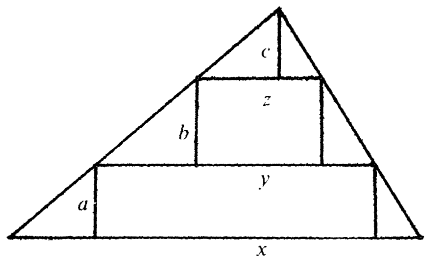
A-2. (29, 15, 31, 14, 38, 11, 2, 15, 6, 4, 21, 15)

Label lengths as in the figure on page 624. Then

$$\frac{A(R) + A(S)}{A(T)} = \frac{ay + bz}{hx/2},$$

where $h = a + b + c$, the altitude of T. By similar triangles,

$$\frac{x}{h} = \frac{y}{b+c} = \frac{z}{c},$$



so

$$\frac{A(R)+A(S)}{A(T)}=\frac{a\dfrac{(b+c)x}{h}+b\dfrac{cx}{h}}{hx/2}=\frac{2}{h^2}(ab+ac+bc).$$

We need to maximize $ab+ac+bc$ subject to $a+b+c=h$. One way to do this is first to fix a, so $b+c=h-a$. Then

$$ab+ac+bc=a(h-a)+bc,$$

and bc is maximized when $b=c$. We now wish to maximize $2ab+b^2$ subject to $a+2b=h$. This is a straight-forward calculus problem giving $a=b=c=h/3$. Hence the maximum ratio is $2/3$ (independent of T).

A-3. (100, 5, 18, 0, 0, 0, 0, 0, 2, 10, 19, 47)

We have $a_n(j+1)+1=(a_n(j)+1)^2$, and hence by induction,

$$a_n(j)+1=(a_n(0)+1)^{2^j}.$$

Therefore

$$\lim_{n\to\infty}a_n(n)=\lim_{n\to\infty}\left(1+\frac{d}{2^n}\right)^{2^n}-1=e^d-1.$$

A-4. (72, 30, 6, 23, 0, 0, 0, 0, 1, 5, 33, 31)

We wish to consider $a_i\equiv 3^{a_{i-1}}\pmod{100}$. Recall that $a^{\phi(n)}\equiv 1\pmod n$ whenever a is relatively prime to n (ϕ is the Euler ϕ-function). Therefore, since $\phi(100)=40$, we can find a_i (mod 100) by knowing a_{i-1} (mod 40). Similarly, we can know a_{i-1} (mod 40) by finding a_{i-2} (mod 16), because $a_{i-1}=3^{a_{i-2}}$ and $\phi(40)=16$. Again, $a_{i-2}=3^{a_{i-3}}$ and $\phi(16)=8$, and therefore

$$a_{i-3}\equiv 3^{a_{i-4}}\equiv 3^{\text{odd integer}}\equiv 3\pmod 8.$$

It follows that $a_{i-2}\equiv 3^3\equiv 11$ (mod 16), and from this,

$$a_{i-1}\equiv 3^{11}\equiv 27\pmod{40},$$

and finally,

$$a_i\equiv 3^{27}\equiv 87\pmod{100}.$$

All of this is valid for all $i\geqslant 4$. Thus, 87 is the only integer that occurs as the last two digits in the decimal expansions of infinitely many a_i.

A-5. (44, 2, 7, 5, 0, 0, 0, 0, 10, 27, 46, 60)
Write

$$I_m = \int_0^{2\pi} \prod_{k=1}^{m} \left(\frac{e^{ikx} + e^{-ikx}}{2} \right) dx = \sum_{\varepsilon_k = \pm 1} \frac{1}{2^m} \int_0^{2\pi} e^{i(\varepsilon_1 + 2\varepsilon_2 + \cdots + m\varepsilon_m)x}\, dx.$$

The integral $\int_0^{2\pi} e^{itx}\, dx$ is zero if t is a nonzero integer and is 2π otherwise. Thus, $I_m \geqslant 0$, and $I_m \neq 0$ if and only if 0 can be written in the form $\varepsilon_1 + 2\varepsilon_2 + \cdots + m\varepsilon_m$ for some $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m \in \{-1,1\}$. For a sum $\varepsilon_1 + 2\varepsilon_2 + \cdots + m\varepsilon_m$, let r denote the sum of the positive terms and s the sum of the absolute values of the negative terms. Then $r - s = m(m+1)/2$. A necessary condition for $r = s$ is that $m(m+1)/2$ be even; that is, that $m \equiv 0$ or $3 \pmod 4$. Thus, the only candidates satisfying these conditions in $1 \leqslant m \leqslant 10$ are $m = 3, 4, 7$, and 8. We find that $I_m \neq 0$ for each of these because $1 + 2 - 3 = 0$, $1 - 2 - 3 + 4 = 0$, $(1 + 2 - 3) + (4 - 5 - 6 + 7) = 0$, and $(1 - 2 - 3 + 4) + (5 - 6 - 7 + 8) = 0$. (Note that this is easy to generalize to all numbers $m \equiv 0$ or $3 \pmod 4$.)

A-6. (8, 2, 0, 0, 0, 0, 0, 0, 3, 19, 22, 147)

Note that $\Gamma(p(x)) = \int_0^1 |p(e(\theta))|^2\, d\theta$, where $e(\theta) = e^{2\pi i\theta}$. Thus,

$$\Gamma(f(x)^n) = \int_0^1 |f(e(\theta))|^{2n}\, d\theta = \int_0^1 |3e(\theta) + 1|^{2n} |e(\theta) + 2|^{2n}\, d\theta.$$

But

$$|e(\theta) + 2| = |e(\theta)|\, |1 + 2e(-\theta)| = |1 + 2e(-\theta)| = |\overline{1 + 2e(\theta)}| = |1 + 2e(\theta)|.$$

Therefore,

$$\Gamma(f(x)^n) = \int_0^1 |3e(\theta) + 1|^{2n} |2e(\theta) + 1|^{2n}\, d\theta = \Gamma(g(x)^n),$$

where we have set $g(x) = 6x^2 + 5x + 1$.

B-1. (112, 29, 0, 0, 0, 0, 0, 0, 0, 30, 13, 17)

Clearly $k > 1$; otherwise $p(x) = x^5$ and $m_1, \ldots, m_5$ are not distinct. Assume $k = 2$, so $p(x) = x^5 + ax^j$ with $0 \leqslant j \leqslant 4$. We can't have $j \geqslant 2$ since then at least two of the m_i's are equal to 0. Hence $p(x) = x^5 + a$ or $p(x) = x(x^4 + a)$ with $a \neq 0$. But $x^5 + a$ and $x^4 + a$ have at most two real zeros. Therefore $k \geqslant 3$.

Set $m_1 = -2$, $m_2 = -1$, $m_3 = 0$, $m_4 = 1$, $m_5 = 2$. Then

$$p(x) = x(x^2 - 1)(x^2 - 4) = x^5 - 5x^3 + 4x.$$

Hence $k = 3$, and this value of k is achieved for the given m_i's.

B-2. (3, 89, 3, 1, 0, 0, 0, 0, 2, 11, 37, 55)

An examination of low order cases leads one to conjecture that $f_n(x) = x(x+n)^{n-1}$. Clearly this guess satisfies $f_0(x) = 1$, $f_n(0) = 0$ for $n \geqslant 1$. Now

$$\begin{aligned} f'_{n+1}(x) &= (x + n + 1)^n + nx(x + n + 1)^{n-1} \\ &= (n+1)(x+1)(x+n+1)^{n-1} \\ &= (n+1) f_n(x+1). \end{aligned}$$

Hence $f_n(x) = x(x+n)^{n-1}$ as guessed. Therefore, $f_{100}(1) = 101^{99}$.

B-3. (95, 15, 15, 11, 0, 0, 0, 0, 5, 3, 23, 34)

Suppose, contrariwise, that $a_{m,n} \leqslant mn$ for all (m, n). Let

$$R(k) = \{(i, j) : a_{i,j} \leqslant k\}.$$

By hypothesis, $|R(k)| = 8k$. On the other hand, $ij \leqslant k$ implies that $(i, j) \in R(k)$ and there are

$$\left\lfloor \frac{k}{1} \right\rfloor + \left\lfloor \frac{k}{2} \right\rfloor + \cdots + \left\lfloor \frac{k}{k} \right\rfloor \geqslant \left(\frac{k}{1} - 1\right) + \cdots + \left(\frac{k}{k} - 1\right) > k(\log k - 1)$$

such pairs. Hence $8k > k(\log k - 1)$, which is a contradiction for $k > e^9$.

B-4. $(115, 30, 9, 1, 2, 6, 4, 0, 9, 8, 11, 6)$

Let $p = (\cos\theta, \sin\theta)$ and $q = (x, y)$. The other two vertices of R are $(\cos\theta, y)$ and $(x, \sin\theta)$, so no point of R lies outside of C if and only if $\cos^2\theta + y^2 \leqslant 1$ and $\sin^2\theta + x^2 \leqslant 1$, or equivalently, $|y| \leqslant |\sin\theta|$ and $|x| \leqslant |\cos\theta|$. Note that these conditions imply that (x, y) lies inside the circle, so that, for any θ, the probability that (x, y) satisfies these conditions is

$$\frac{2|\sin\theta|2|\cos\theta|}{\pi} = \frac{2}{\pi}|\sin 2\theta|,$$

and the overall probability is

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{2}{\pi}|\sin 2\theta|\,d\theta = \frac{1}{2\pi}\cdot\frac{2}{\pi}\cdot 4 = \frac{4}{\pi^2}.$$

B-5. $(14, 8, 2, 2, 0, 0, 0, 0, 2, 2, 61, 110)$

Let $I(x) = \int_0^\infty t^{-1/2}e^{-at-xt^{-1}}\,dt$, where $a = 1985$. Then

$$I'(x) = -\int_0^\infty t^{-1/2}e^{-at-xt^{-1}}t^{-1}\,dt = -\int_0^\infty t^{-3/2}e^{-at-xt^{-1}}\,dt.$$

Make the substitution $u = 1/t$, and the last equation becomes

$$I'(x) = -\int_0^\infty u^{-1/2}e^{-au^{-1}-xu}\,du.$$

Now let $w = \dfrac{x}{a}u$, and the last equation is

$$I'(x) = -\left(\frac{a}{x}\right)^{1/2}\int_0^\infty w^{-1/2}e^{-xw^{-1}-aw}\,dw = -\left(\frac{a}{x}\right)^{1/2}I(x).$$

Therefore, $\log I(x) = -2(ax)^{1/2} + C$, or equivalently, $I(x) = ke^{-2(ax)^{1/2}}$. Also,

$$k = I(0) = \int_0^\infty t^{-1/2}e^{-at}\,dt = \int_0^\infty 2e^{-at^2}\,dt = \frac{\sqrt{\pi}}{\sqrt{a}}.$$

This yields

$$I(a) = \frac{\sqrt{\pi}}{\sqrt{a}}e^{-2a}.$$

(Note: the integral is essentially the K-Bessel function $K_{1/2}(3970)$.)

B-6. $(5, 0, 0, 0, 0, 4, 0, 0, 9, 7, 26, 150)$

Let $S = \sum_{i=1}^r M_i$. For any j, $1 \leqslant j \leqslant r$,

$$M_jS = \sum_{i=1}^r M_jM_i = \sum_{i=1}^r M_i = S, \quad \text{and hence} \quad S^2 = \sum_{j=1}^r M_jS = rS.$$

Therefore the minimal polynomial $p(x)$ for S divides $x^2 - rx$ and every eigenvalue of S is either 0 or r. Since $\operatorname{tr}(S) = 0$, every eigenvalue of S is zero. Every eigenvalue of $S - rI$ is $-r$, and therefore $S - rI$ is invertible. Hence, from $S(S - rI) = 0$, we get $S = 0$.