The William Lowell Putnam Mathematical Competition

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The following results of the forty-seventh William Lowell Putnam Mathematical Competition, held on December 6, 1986, have been determined in accordance with the governing regulations. This annual contest is supported by the William Lowell Putnam Prize Fund for the Promotion of Scholarship, left by Mrs. Putnam in memory of her husband, and is held under the auspices of the Mathematical Association of America.

The first prize, $5,000, was awarded to the Department of Mathematics of Harvard University. The members of the winning team were: Douglas S. Jungreis, Bjorn M. Poonen, and David J. Zuckerman; each was awarded a prize of two hundred fifty dollars.

The second prize, $2,500, was awarded to the Department of Mathematics of Washington University, St. Louis. The members of its team were: Daniel N. Ropp, Dougin A. Walker, and Japheth L. M. Wood; each was awarded a prize of $200.

The third prize, $1,500, was awarded to the Department of Mathematics of the University of California, Berkeley. The members of its team were: Michael J. McGrath, David P. Moulton, and Christopher S. Welty; each was awarded a prize of $150.

The fourth prize, $1,000, was awarded to the Department of Mathematics of Yale University. The members of its team were: Thomas O. Andrews, Kamal F. Khuri-Makdisi, and David R. Steinsaltz; each was awarded a prize of $100.

The fifth prize, $500, was awarded to the Department of Mathematics of the Massachusetts Institute of Technology. The members of its team were: David Blackston, James P. Ferry, and Waldemar P. Horwat; each was awarded a prize of fifty dollars.

The six highest-ranking individual contestants, in alphabetical order, were David J. Grabiner, Princeton University; Waldemar P. Horwat, Massachusetts Institute of Technology; Douglas S. Jungreis, Harvard University; David J. Moews, Harvard University; Bjorn M. Poonen, Harvard University; and David J. Zuckerman, Harvard University. Each of these students was designated a Putnam Fellow by the Mathematical Association of America and awarded a prize of $500 by the Putnam Prize Fund.

The next four highest-ranking individuals, in alphabetical order, were Yong Yao Du, University of Waterloo; Gregory J. Kuperberg, Harvard University; John A. Overdeck, Stanford University; and Michael Reid, Harvard University.

The following teams, named in alphabetical order, received honorable mention: University of British Columbia, with team members Wayne J. Broughton, Marek J. Radzikowski, and Russil Wvong; California Institute of Technology, with team members Leland Brown, Stanley Chen, and Darien G. Lefkowitz; Princeton University, with team members Douglas Davidson, David J. Grabiner, and Randall G. Rose; Rice University, with team members Charles R. Ferenbaugh, Thomas Hyer, and John M. Steinke; and the University of Waterloo, with team members Yong Yao Du, Eric Veech, and Minh Tue Vo.

Honorable mention was achieved by the following thirty-seven individuals named in alphabetical order: Thomas O. Andrews, Yale University; John B. Boyland, University of California, Davis; William Chen, Auburn University; Constantine N. Costes, Harvard University; William P. Cross, California Institute of Technology; Henri R. Darmon, McGill University; Michael B. Davis, Case Western Reserve University; Samuel S. Dooley, Texas A. & M. University; Glenn D. Ellison, Harvard University; Bryan K. Feir, University of Waterloo; Charles R. Ferenbaugh, Rice University; James P. Ferry, Massachusetts Institute of Technology; Patrick T. Headley, Case Western Reserve University; Earl A. Hubbell, California Institute of Technology; William J. Jockusch, Carleton College; Daniel W. Johnson, Rose-Hulman Institute of Technology; Kamal F. Khuri-Makdisi, Yale University; Darien G. Lefkowitz, California Institute of Technology; Michael J. McGrath, University of California, Berkeley; David P. Moulton, University of California, Berkeley; Matthew D. Mullin, Princeton University; Du Nguyen, University of Ottawa; Ravi K. Ramakrishna, Cornell University; Daniel N. Ropp, Washington University, St. Louis; Randall G. Rose, Princeton University; David B. Secrest, University of Illinois, Champaign-Urbana; Kenneth W. Shirriff, University of Waterloo; Stephen A. Smith, University of Waterloo; Theron W. Stanford, California Institute of Technology; John M. Steinke, Rice University; David R. Steinsaltz, Yale University; Constantin T. Teleman, Indiana University, Bloomington; Dougin A. Walker, Washington University, St. Louis; Christopher S. Welty, University of California, Berkeley; Karl M. Westerberg, Carnegie-Mellon University; Japheth L. M. Wood, Washington University, St. Louis; and James C. Yeh, Princeton University.

The other individuals who achieved ranks among the top 96, in alphabetical order of their schools, were: Auburn University, Darren C. Abbott; University of British Columbia, Wayne J. Broughton, Marek J. Radzikowski, Russil Wvong; California Institute of Technology, Leland F. Brown, Stanley Chen, Thomas J. Lenosky; California Polytechnic State University, San Luis Obispo, Daniel L. Krejsa; California State University, Fresno; Chiu Liu; University of California, Berkeley, John Stanley Tillinghast; Carleton University, Serge Elnitsky, Lones A. Smith; Carnegie-Mellon University, Petros I. Hadjicostas, Joseph G. Keane; University of Chicago, Robert M. Beals; Columbia University, Martin J. Strauss, Ali F. Yegulalp; Concordia University, Chinh Mai; University of Connecticut, Anthony J.

Zajac; Harding University, Scott T. Burleson; Harvard University, Chenteh Kenneth Fan, Glen T. Whitney; Massachusetts Institute of Technology, David T. Blackston, Larry Buxbaum, Jordan A. Drachman, Mark Kantrowitz; University of Massachusetts, Amherst, James F. Riordan; University of Missouri, Rolla, David A. Betz; Université de Montréal, François Bedard; University of New Brunswick, Iain G. DeMille; Ohio State University, Thomas E. Barrett; Princeton University, Joseph J. Bohman, Rahul V. Pandharipande; Queen's University, Neale Ginsburg, Krishna Rajagopal; Rice University, Thomas M. Hyer; Stanford University, Thomas H. Chung, Joshua R. Zucker; Texas A. & M. University, Mark G. Yarbrough; University of Texas, El Paso, Luis Gerardo Valdes Sanchez; University of Toronto, Edward J. Doolittle, Jeffrey S. Rosenthal; University of Washington, Seattle, Dean M. Yasuda; University of Waterloo, Christopher K. Anand, Frank M. D'Ippolito, Peter J. Fowler, John S. Omielan, Minh Tue Vo; University of Wisconsin, Madison, Robert C. Mattson.

There were 2094 individual contestants from 358 colleges and universities in Canada and the United States in the competition of December 6, 1986. Teams were entered by 270 institutions.

The Questions Committee for the forty-seventh competition consisted of Richard P. Stanley (Chairman), Abraham P. Hillman, and Harold M. Stark; they composed the problems listed below and were most prominent among those suggesting solutions.

PROBLEMS

Problem A-1. Find, with explanation, the maximum value of $f(x) = x^3 - 3x$ on the set of all real numbers x satisfying $x^4 + 36 \leqslant 13x^2$.

Problem A-2. What is the units (i.e., rightmost) digit of $\left[\dfrac{10^{20000}}{10^{100}+3}\right]$? Here $[x]$ is the greatest integer $\leqslant x$.

Problem A-3. Evaluate $\sum_{n=0}^{\infty}\text{Arccot}(n^2+n+1)$, where $\text{Arccot}\, t$ for $t \geqslant 0$ denotes the number θ in the interval $0 < \theta \leqslant \pi/2$ with $\cot\theta = t$.

Problem A-4. A *transversal* of an $n \times n$ matrix A consists of n entries of A, no two in the same row or column. Let $f(n)$ be the number of $n \times n$ matrices A satisfying the following two conditions:

(a) Each entry $\alpha_{i,j}$ of A is in the set $\{-1, 0, 1\}$.

(b) The sum of the n entries of a transversal is the same for all transversals of A.

An example of such a matrix A is

$$A = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Determine with proof a formula for $f(n)$ of the form

$$f(n) = a_1 b_1^n + a_2 b_2^n + a_3 b_3^n + a_4,$$

where the a_i's and b_i's are rational numbers.

Problem A-5. Suppose $f_1(x), f_2(x), \dots, f_n(x)$ are functions of n real variables $x = (x_1, \dots, x_n)$ with continuous second-order partial derivatives everywhere on R^n. Suppose further that there are constants c_{ij} such that

$$\frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i} = c_{ij}$$

for all i and j, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$. Prove that there is a function $g(x)$ on R^n such that $f_i + \partial g/\partial x_i$ is linear for all i, $1 \leqslant i \leqslant n$. (A linear function is one of the form

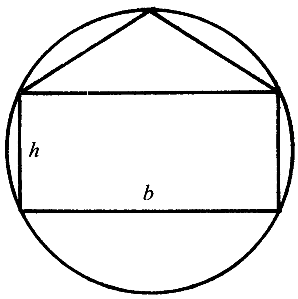
$$a_0 + a_1 x_1 + a_2 x_2 + \cdots + a_n x_n.)$$

Problem A-6. Let $a_1, a_2, \dots, a_n$ be real numbers, and let $b_1, b_2, \dots, b_n$ be distinct positive integers. Suppose there is a polynomial $f(x)$ satisfying the identity

$$(1-x)^n f(x) = 1 + \sum_{i=1}^{n} a_i x^{b_i}.$$

Find a simple expression (not involving any sums) for $f(1)$ in terms of $b_1, b_2, \dots, b_n$ and n (but independent of $a_1, a_2, \dots, a_n$).

Problem B-1. Inscribe a rectangle of base b and height h and an isosceles triangle of base b in a circle of radius one as shown.
For what value of h do the rectangle and triangle have the same area?



Problem B-2. Prove that there are only a finite number of possibilities for the ordered triple $T = (x - y, y - z, z - x)$ where x, y, and z are complex numbers satisfying the simultaneous equations

$$x(x-1) + 2yz = y(y-1) + 2zx = z(z-1) + 2xy,$$

and list all such triples T.

Problem B-3. Let Γ consist of all polynomials in x with integer coefficients. For f and g in Γ and m a positive integer, let $f \equiv g \pmod m$ mean that every coefficient of $f - g$ is an integral multiple of m. Let n and p be positive integers with p prime. Given that f, g, h, r, and s are in Γ with $rf + sg \equiv 1 \pmod p$ and $fg \equiv h \pmod p$, prove that there exist F and G in Γ with $F \equiv f \pmod p$, $G \equiv g \pmod p$, and $FG \equiv h \pmod{p^n}$.

Problem B-4. For a positive real number r, let $G(r)$ be the minimum value of $|r - \sqrt{m^2 + 2n^2}\,|$ for all integers m and n. Prove or disprove the assertion that $\lim_{r \to \infty} G(r)$ exists and equals 0.

Problem B-5. Let $f(x, y, z) = x^2 + y^2 + z^2 + xyz$. Let $p(x, y, z)$, $q(x, y, z)$, $r(x, y, z)$ be polynomials with real coefficients satisfying

$$f(p(x, y, z), q(x, y, z), r(x, y, z)) = f(x, y, z).$$

Prove or disprove the assertion that the sequence p, q, r consists of some permutation of $\pm x$, $\pm y$, $\pm z$, where the number of minus signs is 0 or 2.

Problem B-6. Suppose A, B, C, D are $n \times n$ matrices with entries in a field F, satisfying the conditions that AB^t and CD^t are symmetric and $AD^t - BC^t = I$. Here I is the $n \times n$ identity matrix, and if M is an $n \times n$ matrix, M^t is the transpose of M. Prove that $A^tD - C^tB = I$.

SOLUTIONS

In the 12-tuples $(n_{10}, n_9, \ldots, n_0, n_{-1})$ following each problem number below, n_i for $10 \geq i \geq 0$ is the number of students among the top 201 contestants achieving i points for the problem and n_{-1} is the number of those not submitting solutions.

A-1 (152, 23, 10, 7, 0, 0, 0, 2, 2, 3, 1, 1)

The condition that $x^4 + 36 \leqslant 13x^2$ is equivalent to

$$(x - 3)(x - 2)(x + 2)(x + 3) \leqslant 0.$$

The latter is satisfied if and only if x is in the closed interval $[-3, -2]$ or the closed interval $[2, 3]$. The function f is increasing on these intervals because for such x, $f'(x) = 3(x^2 - 1) > 0$. It follows that the maximum value of f over this domain is $\max\{f(-2), f(3)\} = 18$.

A-2 (155, 0, 0, 0, 0, 0, 0, 0, 0, 0, 33, 13)

The greatest integer is

$$I = \frac{10^{20000} - 3^{200}}{10^{100} + 3}$$

since the remainder is

$$\frac{3^{200}}{10^{100} + 3} < 1.$$

We have

$$I \equiv \frac{-3^{200}}{3} \pmod{10} \equiv -3^{199} \pmod{10} \equiv -3^3(3^4)^{49} \pmod{10} \equiv -27 \pmod{10}$$
$$\equiv 3 \pmod{10}.$$

The last digit is 3.

A-3 (53, 6, 15, 1, 0, 0, 0, 1, 12, 1, 26, 86)
Using

$$\cot(\alpha-\beta) = \frac{\cot\alpha\cot\beta+1}{\cot\beta-\cot\alpha},$$

one sees that $\operatorname{Arccot}(1+n+n^2) = \operatorname{Arccot} n - \operatorname{Arccot}(n+1)$. Then the series telescopes to $\lim_{n\to\infty}(\operatorname{Arccot} 0 - \operatorname{Arccot}(n+1)) = \pi/2$.

A-4 (21, 3, 4, 4, 5, 7, 0, 6, 3, 7, 23, 118)
We first prove:

LEMMA. *If an $n\times n$ matrix (α_{ij}) satisfies (b), then there are unique numbers $c_1 = 0, c_2, \dots, c_n, d_1, d_2, \dots, d_n$ such that $\alpha_{ij} = c_i + d_j$. Conversely, any such choice of c_i's and d_j's yields a unique matrix (α_{ij}) satisfying (b).*

Proof. If $\alpha_{ij} = c_i + d_j$ then any transversal of A sums to

$$\sum_{i=1}^{n} c_i + \sum_{j=1}^{n} d_j,$$

so (b) is satisfied.

Conversely, suppose (α_{ij}) satisfies (b). Define $d_j = \alpha_{1j}$ and $c_i = \alpha_{i1} - d_1 = \alpha_{i1} - \alpha_{1j}$ (so $c_1 = 0$). Since (b) is satisfied, we have $\alpha_{ij} + \alpha_{11} = \alpha_{i1} + \alpha_{1j}$, so $\alpha_{ij} = \alpha_{i1} + \alpha_{1j} - \alpha_{11} = (c_i + d_1) + d_j - d_1 = c_i + d_j$, as desired.

The c_i's and d_j's are unique since $c_1 = 0$ and $\alpha_{1j} = c_1 + d_j$ forces $d_j = \alpha_{1j}$, and then $\alpha_{i1} = c_i + d_1$ forces $c_i = \alpha_{i1} - d_1$. This proves the lemma.

Thus $f(n)$ is equal to the number of $2n$-tuples $(c_1 = 0, c_2, \dots, c_n, d_1, \dots, d_n)$ for which $c_i + d_j = 0, \pm 1$. We break the possibilities into the following eight cases.

distinct c_i's	possible d_i's	number of c_i's	number of d_i's	product
0	$0, -1, 1$	1	3^n	3^n
$0, -2$	1	$2^{n-1}-1$	1	$2^{n-1}-1$
$0, -2, -1$	1	$3^{n-1}-2^n+1$	1	$3^{n-1}-2^n+1$
$0, 2$	-1	$2^{n-1}-1$	1	$2^{n-1}-1$
$0, 1, 2$	-1	$3^{n-1}-2^n+1$	1	$3^{n-1}-2^n+1$
$0, 1$	$0, -1$	$2^{n-1}-1$	2^n	$\frac{1}{2}4^n-2^n$
$0, -1$	$0, 1$	$2^{n-1}-1$	2^n	$\frac{1}{2}4^n-2^n$
$0, -1, 1$	0	$3^{n-1}-2^n+1$	1	$3^{n-1}-2^n+1$

Summing the last column gives

$$f(n) = 4^n + 2 \cdot 3^n - 4 \cdot 2^n + 1.$$

A-5 (13, 4, 0, 0, 0, 0, 0, 1, 0, 2, 39, 142)

Note that $c_{ji} = -c_{ij}$ for all i and j. Let $h_i = \frac{1}{2}\Sigma_j c_{ij} x_j$ so that $\partial h_i / \partial x_j = \frac{1}{2} c_{ij}$. Then

$$\frac{\partial h_i}{\partial x_j} - \frac{\partial h_j}{\partial x_i} = \frac{1}{2} c_{ij} - \frac{1}{2} c_{ji} = c_{ij} = \frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i}$$

and so

$$\frac{\partial (h_i - f_i)}{\partial x_j} = \frac{\partial (h_j - f_j)}{\partial x_i}$$

for all i and j. Hence $(h_1 - f_1, \ldots, h_n - f_n)$ is a gradient and so there is a function g such that $\partial g / \partial x_i = h_i - f_i$. In other words, $f_i + \partial g / \partial x_i = h_i$ is linear.

A-6 (1, 4, 1, 1, 0, 1, 0, 0, 6, 4, 64, 119)

Write $(b)_j = b(b-1)\ldots(b-j+1)$. Differentiating $(1-x)^n f(x) = 1 + \Sigma_{i=1}^n a_i x^{b_i}$ j times $(0 \leqslant j \leqslant n)$ and putting $x = 1$ yields

$$\begin{aligned} 0 &= 1 + \sum a_i, \\ 0 &= \sum a_i b_i, \\ 0 &= \sum a_i (b_i)_2, \\ &\vdots \\ 0 &= \sum a_i (b_i)_{n-1}, \\ n! f(1) &= \sum a_i (b_i)_n. \end{aligned}$$

Solve the first n equations for $a_1, \ldots, a_n$ by Cramer's rule and substitute into the last equation. We get

$$n! f(1) = \frac{\sum_{i=1}^{n} (b_i)_n (-1)^i \begin{vmatrix} b_1 & \cdots & \hat{b}_i & \cdots & b_n \\ (b_1)_{n-1} & \cdots & (\hat{b}_i)_{n-1} & \cdots & (b_n)_{n-1} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ b_1 & b_2 & & b_n \\ & & \vdots & \\ (b_1)_{n-1} & (b_2)_{n-1} & \cdots & (b_n)_{n-1} \end{vmatrix}}$$

where indicates a missing entry. The denominator D is a polynomial of total degree $\binom{n}{2}$ in $b_1, \ldots, b_n$ and vanishes whenever $b_i = b_j$, $i \neq j$. Hence, $D =$

$C\prod_{i<j}(b_i - b_j)$. The constant C is seen to be 1 by considering, say, the coefficient of $b_2 b_3^2 \cdots b_n^{n-1}$, so $D = \prod_{i<j}(b_i - b_j)$. (This can also be proved by performing elementary row operations on the Vandermonde determinant.)

Put $b_i = b_j$ $(i \neq j)$ in the numerator. All terms vanish except two, which are of equal magnitude but opposite sign. Hence the numerator is divisible by $b_i - b_j$ and thus by the denominator. Therefore $n!f(1)$ is a polynomial in $b_1, \ldots, b_n$. The degree of this polynomial is $\leqslant n$, since each term in the numerator has degree n more than the degree $\binom{n}{2}$ of the denominator.

Now put $b_i = 0$. The denominator doesn't vanish, but each term in the numerator does. Thus, $n!f(1)$ is divisible by b_i, so $n!f(1) = kb_1b_2 \cdots b_n$ for some constant k. By putting $f(x) = 1$ (so $b_i = i$), we see $k = 1$. Thus $f(1) = b_1 \cdots b_n/n!$.

B-1 (183, 3, 7, 0, 0, 0, 0, 0, 4, 2, 1, 1)

The altitude of the triangle is $\frac{1}{2}(2 - h)$. Equal area means

$$h = \frac{1}{2}(\text{altitude of triangle}) = \frac{1}{4}(2 - h),$$

so $h = 2/5$.

B-2 (123, 31, 16, 3, 0, 0, 0, 0, 16, 5, 2, 5)

The system is equivalent to $0 = (x - y)(x + y - 1 - 2z) = (y - z)(y + z - 1 - 2x) = (z - x)(z + x - 1 - 2y)$. If no two of x, y, z are equal, then $x + y - 1 - 2z = y + z - 1 - 2x = z + x - 1 - 2y = 0$. Adding these gives the contradiction $-3 = 0$. So at least two of x, y, z are equal. If $x = y$ and $y \neq z$, then $z = 2x + 1 - y = x + 1$. In this case we find that $x - y = 0$, $y - z = -1$, and $z - x = 1$. A similar result follows when $y = z$ and $z \neq x$, and when $z = x$ and $x \neq y$. Thus, the only possibilities for $(x - y, y - z, z - x)$ are $(0, 0, 0)$, $(0, -1, 1)$, $(1, 0, -1)$, and $(-1, 1, 0)$. One shows easily that each of these actually occurs.

B-3 (26, 5, 4, 1, 0, 1, 0, 4, 3, 5, 33, 119)

Suppose for $k \geqslant 1$ that we have polynomials F_k and G_k with integer coefficients such that $F_k \equiv f \pmod{p}$, $G_k \equiv g \pmod{p}$, and $F_kG_k \equiv h \pmod{p^k}$. For $k = 1$ this can be done with $F_1 = f$, $G_1 = g$. Let $h - F_kG_k = tp^k$ for some $t \in \Gamma$. Let $F_{k+1} = F_k + stp^k$ and $G_{k+1} = G_k + rtp^k$. Then $F_{k+1} \equiv F_k \equiv f \pmod{p}$ and $G_{k+1} \equiv G_k \equiv g \pmod{p}$, and

$$\begin{aligned} F_{k+1}G_{k+1} &= F_kG_k + tp^k(rF_k + sG_k) + rst^2p^{2k} \\ &\equiv F_kG_k + tp^k(rF_k + sG_k) \pmod{p^{k+1}}. \end{aligned}$$

By hypothesis, $rF_k + sG_k \equiv rf + sg \equiv 1 \pmod{p}$, and so $rF_k + sG_k = 1 + qp$ for some $q \in \Gamma$. Thus

$$\begin{aligned} F_{k+1}G_{k+1} &\equiv F_kG_k + tp^k(1 + qp) \pmod{p^{k+1}} \\ &\equiv F_kG_k + tp^k \pmod{p^{k+1}} \\ &\equiv h \pmod{p^{k+1}}, \end{aligned}$$

and we are done by induction. (The result holds whether or not p is a prime and is independent of the number of variables in the polynomials.)

B-4 (22, 8, 6, 6, 0, 0, 0, 0, 4, 7, 59, 89)

Let m be the largest integer in $N = \{0, 1, \dots\}$ with $r^2 \geqslant m^2$. Let n be the largest integer in N with $(r^2 - m^2)/2 \geqslant n^2$. It follows that $r^2 - m^2 < 2m + 1 \leqslant 2r + 1$ and that

$$\begin{aligned} \frac{r^2 - m^2}{2} - n^2 < 2n + 1 &\leqslant 2\sqrt{\frac{r^2 - m^2}{2}} + 1 \\ &< 2\sqrt{\frac{2r + 1}{2}} + 1 = \sqrt{2(2r + 1)} + 1. \end{aligned}$$

Hence $r^2 - m^2 - 2n^2 < 2\sqrt{2}\sqrt{2r + 1} + 2$. Since

$$r^2 - m^2 - 2n^2 = \left(r - \sqrt{m^2 + 2n^2}\right)\left(r + \sqrt{m^2 + 2n^2}\right),$$

$$r - \sqrt{m^2 + 2n^2} = \frac{r^2 - m^2 - 2n^2}{r + \sqrt{m^2 + 2n^2}} < \frac{2\sqrt{2}\sqrt{2r + 1} + 2}{r},$$

which $\to 0$ as $r \to \infty$. Hence

$$\lim_{r \to \infty} G(r) = \lim_{r \to \infty} \left| r - \sqrt{m^2 + 2n^2} \right| = 0.$$

B-5 (10, 0, 0, 0, 0, 0, 0, 0, 0, 0, 58, 133)

The assertion is false. Take $p = x$, $q = y$, $r = -z - xy$. We can generate an infinite sequence of counterexamples by noting that if (p, q, r) is a solution then so is $(p, q, -r - pq)$.

B-6 (3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 32, 166)

The conditions of the problem are

(i) $AB^t = (AB^t)^t = BA^t$,

(ii) $CD^t = (CD^t)^t = DC^t$,

(iii) $AD^t - BC^t = I$.

Condition (i) implies $BA^t - AB^t = 0$ (the $n \times n$ zero matrix). Condition (ii) implies $CD^t - DC^t = 0$, and the transpose of condition (iii) is $DA^t - CB^t = I^t = I$. Hence we have

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix} = \begin{pmatrix} AD^t - BC^t & -AB^t + BA^t \\ CD^t - DC^t & -CB^t + DA^t \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

From this it follows that

$$\begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

as well, and the lower right-hand corner of this is $-C^tB + A^tD = I$.