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The following results of the forty-eighth William Lowell Putnam Mathematical Competition, held on December 5, 1987, have been determined in accordance with the governing regulations. This annual contest is supported by the William Lowell Putnam Prize Fund for the Promotion of Scholarship, left by Mrs. Putnam in memory of her husband, and is held under the auspices of the Mathematical Association of America.

The first prize, $5,000, was awarded to the Department of Mathematics of Harvard University. The members of the winning team were: David J. Moews, Bjorn M. Poonen, and Michael Reid; each was awarded a prize of $250.

The second prize, $2,500, was awarded to the Department of Mathematics of Princeton University. The members of the winning team were: Daniel J. Bernstein, David J. Grabiner, and Matthew D. Mullin; each was awarded a prize of $200.

The third prize, $1,500, was awarded to the Department of Mathematics of Carnegie-Mellon University. The members of the winning team were: Petros I. Hadjicostas, Joseph G. Keane, and Karl M. Westerberg; each was awarded a prize of $150.

The fourth prize, $1,000, was awarded to the Department of Mathematics of the University of California, Berkeley. The members of the winning team were: David P. Moulton, Jonathan E. Shapiro, and Christopher S. Welty; each was awarded a prize of $100.

The fifth prize, $500, was awarded to the Department of Mathematics of the Massachusetts Institute of Technology. The members of the winning team were: David T. Blackston, James P. Ferry, and Waldemar P. Horwat; each was awarded a prize of $50.

The six highest-ranking individual contestants, in alphabetical order, were David J. Grabiner, Princeton University; David J. Moews, Harvard University; Bjorn M. Poonen, Harvard University; Michael Reid, Harvard University; Constantin S. Teleman, Harvard University; and John S. Tillinghast, University of California, Davis. Each of these students was designated a Putnam Fellow by the Mathematical Association of America and awarded a prize of $500 by the Putnam Prize Fund.

The next five highest-ranking individuals, in alphabetical order, were Daniel J. Bernstein, Princeton University; Constantine N. Costes, Harvard University; Jeremy A. Kahn, Harvard University; Rav Kumar Ramakrishna, Cornell University; and Japheth Wood, Washington University, St. Louis. Each was awarded a prize of $250.

The following teams, named in alphabetical order, received honorable mention: Rice University, with team members Charles R. Ferenbaugh, Thomas M. Hyer, and John W. McIntosh; Stanford University, with team members Thomas H. Chung, John A. Overdeck, and Joshua R. Zucker; the University of Toronto, with team members Gary F. Baumgartner, Edward I. Doolittle, and Jeffrey S. Rosenthal;

Washington University, St. Louis, with team members Daniel N. Ropp, Peter Shawhan, and Japheth Wood; and Yale University, with team members Kamal F. Khuri-Makdisi, Robert S. Manning, and William M. Nelson.

Honorable mention was achieved by the following thirty-eight individuals named in alphabetical order: Thomas R. Amoth, Oregon State University; Leland F. Brown, California Institute of Technology; Emory F. Bunn, Princeton University; Yeow Meng Chee, University of Waterloo; Timothy Y. Chow, Princeton University; William P. Cross, California Institute of Technology; Galia D. Dafni, Pennsylvania State University; Douglas R. Davidson, Princeton University; Frank M. D'Ippolito, University of Waterloo; Edward J. Doolittle, University of Toronto; Serge Elnitsky, Carleton University; Chenteh Kenneth Fan, Harvard University; Petros Hadjicostas, Carnegie-Mellon University; Thomas R. Hagedorn, Princeton University; Thomas S. Harke, University of Alberta; Waldemar P. Horwat, Massachusetts Institute of Technology; William C. Jockusch, Carleton College; Alex T. Kachura, University of Waterloo; Joseph G. Keane, Carnegie-Mellon University; Kamal F. Khuri-Makdisi, Yale University; Jordan Lampe, University of California, Berkeley; Daniel D. Lee, Harvard University; John W. McIntosh, Rice University; Robert S. Manning, Yale University; David P. Moulton, University of California, Berkeley; Matthew D. Mullin, Princeton University; Du Nguyen, Ottawa University; David L. Petry, University of Oregon; Daniel N. Ropp, Washington University, St. Louis; David B. Secrest, University of Illinois, Urbana-Champaign; Robert G. Southworth, California Institute of Technology; Glenn P. Tesler, California Institute of Technology; Martin Trudeau, Université de Montréal; Christopher S. Welty, University of California, Berkeley; Karl M. Westerberg, Carnegie-Mellon University; Glen T. Whitney, Harvard University; Russil Wvong, University of British Columbia; and Joshua R. Zucker, Stanford University.

The other individuals who achieved ranks among the top 101, in alphabetical order of their schools, were: Amherst College, Peter H. Anspach; Bethel College, Jonathan P. McCammond; University of British Columbia, Wayne J. Broughton; Brown University, Kevin S. McFarland, David J. Morin; California Institute of Technology, Jared C. Bronski, Philip W. Nabours; University of California, Berkeley, Jonathan E. Shapiro; University of California, Los Angeles, Joseph M. Rojas; University of California, San Diego, David L. Ruhm; Carleton University, Michael J. Bradley, Stephen A. Smith; Case Western Reserve University, Patrick T. Headley, William E. Kirby; University of Chicago, Linda E. Green, Robert P. Stingley, Andrew S. Yeh; Dalhousie University, Daniel J. Peters; Harvard University, Michael J. Callahan, David Cook, Michael P. Mitzenmacher; University of Hawaii, Jor-Kuen E. Lo; University of Houston (Clear Lake), John Ken Burton, Jr.; University of Illinois, Urbana-Champaign, James M. Grochocinski; University of Maryland, Catonsville, Michael J. Johnson; Massachusetts Institute of Technology, David T. Blackston, Claudio C. Chamon, James P. Ferry, Mark Kantrowitz, James R. Rauen; University of Michigan, Ann Arbor, William F. Doran IV, Matthew A. Klimesh, Paul Kominsky; Oberlin College, David B. Carlton; University of Pennsylvania, Michael Albert; Princeton University, Rahul V. Pandharipande; University of Regina, Simon H. Lee; Rice University, Charles R. Ferenbaugh, Thomas M. Hyer; Siena College, Gregory S. Spradlin; University of South Carolina, Roger B. Milne; Stanford University, John C. Loftin, John A. Overdeck; University of Texas, Austin, Jared L. Levy; University of Toronto, Gary F. Baumgartner, Jeffrey S.

Rosenthal; Virginia Polytechnic Institute and State University, Patrick R. Brown; Washington and Lee University, John D. Boller; Washington University, St. Louis, David S. Shobe; University of Waterloo, Marc A. Chamberland, Bryan K. Feir, Giuseppe Russo.

There were 2170 individual contestants from 359 colleges and universities in Canada and the United States in the competition of December 5, 1987. Teams were entered by 277 institutions.

The Questions Committee for the forty-eighth competition consisted of Harold M. Stark (Chairman), Gerald A. Heuer, and Abraham P. Hillman; they composed the problems listed below and were most prominent among those suggesting solutions.

PROBLEMS

Problem A-1

Curves A, B, C, and D, are defined in the plane as follows:

$$A = \left\{(x, y)\colon x^2 - y^2 = \frac{x}{x^2 + y^2}\right\},$$

$$B = \left\{(x, y)\colon 2xy + \frac{y}{x^2 + y^2} = 3\right\},$$

$$C = \left\{(x, y)\colon x^3 - 3xy^2 + 3y = 1\right\},$$

$$D = \left\{(x, y)\colon 3x^2y - 3x - y^3 = 0\right\}.$$

Prove that $A \cap B = C \cap D$.

Problem A-2

The sequence of digits

$$1\,2\,3\,4\,5\,6\,7\,8\,9\,1\,0\,1\,1\,1\,2\,1\,3\,1\,4\,1\,5\,1\,6\,1\,7\,1\,8\,1\,9\,2\,0\,2\,1\ldots$$

is obtained by writing the positive integers in order. If the 10^nth digit in this sequence occurs in the part of the sequence in which the m-digit numbers are placed, define $f(n)$ to be m. For example, $f(2) = 2$ because the 100th digit enters the sequence in the placement of the two-digit integer 55. Find, with proof, $f(1987)$.

Problem A-3

For all real x, the real-valued function $y = f(x)$ satisfies

$$y'' - 2y' + y = 2e^x.$$

(a) If $f(x) > 0$ for all real x, must $f'(x) > 0$ for all real x? Explain.
(b) If $f'(x) > 0$ for all real x, must $f(x) > 0$ for all real x? Explain.

Problem A-4

Let P be a polynomial, with real coefficients, in three variables and F be a function of two variables such that

$$P(ux, uy, uz) = u^2F(y - x, z - x) \quad \text{for all real } x, y, z, u,$$

and such that $P(1,0,0) = 4$, $P(0,1,0) = 5$, and $P(0,0,1) = 6$. Also let A, B, C be complex numbers with $P(A, B, C) = 0$ and $|B - A| = 10$. Find $|C - A|$.

Problem A-5

Let

$$\vec{G}(x, y) = \left(\frac{-y}{x^2 + 4y^2}, \frac{x}{x^2 + 4y^2}, 0 \right).$$

Prove or disprove that there is a vector-valued function

$$\vec{F}(x, y, z) = (M(x, y, z), N(x, y, z), P(x, y, z))$$

with the following properties:

(i) M, N, P have continuous partial derivatives for all $(x, y, z) \neq (0,0,0)$;
(ii) Curl $\vec{F} = \vec{0}$ for all $(x, y, z) \neq (0,0,0)$;
(iii) $\vec{F}(x, y, 0) = \vec{G}(x, y)$.

Problem A-6

For each positive integer n, let $a(n)$ be the number of zeros in the base 3 representation of n. For which positive real numbers x does the series

$$\sum_{n=1}^{\infty} \frac{x^{a(n)}}{n^3}$$

converge?

Problem B-1

Evaluate

$$\int_2^4 \frac{\sqrt{\ln(9-x)}\,dx}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}}.$$

Problem B-2

Let r, s, and t be integers with $0 \leqslant r$, $0 \leqslant s$, and $r + s \leqslant t$. Prove that

$$\frac{\binom{s}{0}}{\binom{t}{r}} + \frac{\binom{s}{1}}{\binom{t}{r+1}} + \frac{\binom{s}{2}}{\binom{t}{r+2}} + \cdots + \frac{\binom{s}{s}}{\binom{t}{r+s}} = \frac{t+1}{(t+1-s)\binom{t-s}{r}}.$$

(Note: $\binom{n}{k}$ denotes the binomial coefficient $n(n-1)\cdots(n+1-k)/k(k-1)\cdots 3\cdot 2\cdot 1$.)

Problem B-3

Let F be a field in which $1 + 1 \neq 0$. Show that the set of solutions to the equation $x^2 + y^2 = 1$ with x and y in F is given by $(x, y) = (1,0)$ and

$$(x, y) = \left(\frac{r^2 - 1}{r^2 + 1}, \frac{2r}{r^2 + 1} \right),$$

where r runs through the elements of F such that $r^2 \neq -1$.

Problem B-4

Let $(x_1, y_1) = (0.8, 0.6)$ and let $x_{n+1} = x_n \cos y_n - y_n \sin y_n$ and $y_{n+1} = x_n \sin y_n + y_n \cos y_n$ for $n = 1, 2, 3, \ldots$. For each of $\lim_{n \to \infty} x_n$ and $\lim_{n \to \infty} y_n$, prove that the limit exists and find it or prove that the limit does not exist.

Problem B-5

Let O_n be the n-dimensional zero vector $(0,0,\dots,0)$. Let M be a $2n \times n$ matrix of complex numbers such that whenever $(z_1, z_2, \dots, z_{2n})M = O_n$, with complex z_i, not all zero, then at least one of the z_i is not real. Prove that for arbitrary real numbers $r_1, r_2, \dots, r_{2n}$, there are complex numbers $w_1, w_2, \dots, w_n$ such that

$$\operatorname{Re}\left[M\begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}\right] = \begin{pmatrix} r_1 \\ \vdots \\ r_{2n} \end{pmatrix}$$

(Note: If C is a matrix of complex numbers, $\operatorname{Re}(C)$ is the matrix whose entries are the real parts of entries of C.)

Problem B-6

Let F be the field of p^2 elements where p is an odd prime. Suppose S is a set of $(p^2-1)/2$ distinct nonzero elements of F with the property that for each $a \neq 0$ in F, exactly one of a and $-a$ is in S. Let N be the number of elements in the intersection $S \cap \{2a\colon a \in S\}$. Prove that N is even.

SOLUTIONS

In the 12-tuples $(n_{10}, n_9, \dots, n_0, n_{-1})$ following each problem number below, n_i for $10 \geq i \geq 0$ is the number of students among the top 204 contestants achieving i points for the problem and n_{-1} is the number of those not submitting solutions.

A-1. (72, 24, 22, 0, 0, 0, 0, 0, 5, 0, 45, 36)

Solution 1. First note that $(0,0)$ doesn't belong to either $A \cap B$ or $C \cap D$, so in what follows suppose that $(x, y) \neq (0,0)$.

Let $Eq(i)$, $i = 1,2,3,4$, denote the equation that defines the set A, B, C, D respectively. Also, let $f(x, y)Eq(i)$ denote the equation obtained by multiplying each side of $Eq(i)$ by $f(x, y)$.

The matrix product

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix}\begin{pmatrix} Eq(1) \\ Eq(2) \end{pmatrix} = \begin{pmatrix} Eq(3) \\ Eq(4) \end{pmatrix}$$

shows that $A \cap B \subseteq C \cap D$, and

$$\begin{pmatrix} Eq(1) \\ Eq(2) \end{pmatrix} = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}^{-1}\begin{pmatrix} Eq(3) \\ Eq(4) \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{x}{x^2+y^2} & \dfrac{y}{x^2+y^2} \\ \dfrac{-y}{x^2+y^2} & \dfrac{x}{x^2+y^2} \end{pmatrix}\begin{pmatrix} Eq(3) \\ Eq(4) \end{pmatrix}$$

shows that $C \cap D \subseteq A \cap B$.

These two inclusions show that $A \cap B = C \cap D$.

Solution 2. Let $z = x + iy$. Then

$$\begin{aligned}(x, y) \in A \cap B \quad & \text{iff } z^2 = 3i + 1/z \\ & \text{iff } z^3 = 3iz + 1 \\ & \text{iff } (x, y) \in C \cap D.\end{aligned}$$

A-2. (117, 28, 22, 2, 0, 0, 0, 3, 10, 5, 10, 7)

The r-digit numbers run from 10^{r-1} to $10^r - 1$, so there are $10^r - 10^{r-1}$ of them. Thus, the total number of digits in numbers with at most r digits is $g(r) = \sum_{k=1}^{r} k(10^k - 10^{k-1}) = -1 + \sum_{k=1}^{r-1}(k - (k+1))10^k + r10^r = -\sum_{k=0}^{r-1} 10^k + r10^r = r10^r - (10^r - 1)/9$ for $r \geqslant 1$. But $0 < (10^r - 1)/9 < 10^r$, so $(r-1)10^r < g(r) < r10^r$. Thus, $g(1983) < 1983 \cdot 10^{1983} < 10^4 \cdot 10^{1983} = 10^{1987}$, and $g(1984) > 1983 \cdot 10^{1984} > 10^3 \cdot 10^{1984} = 10^{1987}$. It follows that $f(1987) = 1984$.

A-3. (119, 32, 1, 0, 1, 4, 2, 0, 0, 5, 11, 29)

The general solution to the differential equation is $f(x) = (x^2 + bx + c)e^x$ with b and c real. For such a function $f'(x) = (x^2 + (b+2)x + (b+c))e^x$. Clearly $f(x) > 0$ for all x if and only if $D = b^2 - 4ac < 0$, and $f'(x) > 0$ for all x if and only if $D' = (b+2)^2 - 4(b+c) = b^2 + 4b + 4 - 4b - 4c = D + 4 < 0$. The answer to (a) is "no" because $D < 0$ does not imply $D' < 0$. (For example, take $b = c = 1$; then $f(x) > 0$ for all x but $f'(-1) = 0$.) The answer to (b) is "yes" because $D' < 0$ implies $D < 0$.

A-4. (14, 8, 6, 4, 0, 0, 0, 1, 28, 9, 25, 109)

Letting $u = 1$ and $x = 0$, we have $F(y, z) = P(0, y, z)$ is a polynomial. $F(uy, uz) = P(0, uy, uz) = u^2F(y, z)$, so F is homogeneous of degree 2. Now $P(x, y, z) = F(y - x, z - x)$ implies that

$$P(x, y, z) = a(y - x)^2 + b(y - x)(z - x) + c(z - x)^2$$

with a, b, c real. Then $4 = P(1,0,0) = a + b + c$, $5 = P(0,1,0) = a$, and $6 = P(0,0,1) = c$. It follows that $4 = a + b + c = 5 + b + 6$, and so $b = -7$.

For the complex numbers A, B, C of the hypothesis, we have $5(B - A)^2 - 7(B - A)(C - A) + 6(C - A)^2 = 0$. Let $m = (C - A)/(B - A)$. Then $5 - 7m + 6m^2 = 0$. The roots of $6m^2 - 7m + 5 = 0$ are complex, so $|m| = \sqrt{5/6}$. Hence, $|C - A| = \sqrt{5/6}\,|B - A| = (5/3)\sqrt{30}$.

A-5. (8, 0, 2, 0, 0, 0, 0, 0, 1, 1, 57, 135)

Note that Curl $\vec{G} = \vec{0}$ unless (x, y, z) is on the z-axis. If $\vec{F}$ exists, then by Stokes' Theorem,

$$\int_C \vec{G}(x, y) \cdot d\vec{r} = \int_C \vec{F}(x, y, z) \cdot d\vec{r} = \int\!\!\int_S (\text{Curl } \vec{F}) \cdot \vec{n}\, dS = 0,$$

where C is the ellipse $x^2 + 4y^2 = 1$, $z = 0$, in the xy-plane and S is the part of the ellipsoid $x^2 + 4y^2 + z^2 = 1$ with $z \geqslant 0$. However, the integral is not zero. For example, on C, $x^2 + 4y^2 = 1$ and thus,

$$\int_C \vec{G}(x, y) \cdot d\vec{r} = \int_C (-y\vec{i} + x\vec{j}\,) \cdot d\vec{r} = \int\!\!\int_E 2\, dx\, dy = 2\,\text{Area}(E),$$

where E is the interior of C. Thus $\vec{F}$ does not exist.

A-6. (5, 10, 2, 0, 1, 0, 0, 3, 1, 4, 80, 98)

For each integer $k \geqslant 0$, the integer n in base 3 has $k+1$ digits iff $3^k \leqslant n < 3^{k+1} - 1$. Among the integers in this interval there are $\binom{k}{i} 2^{k+1-i}$ for which $a(n) = i$, so

$$\sum_{n=3^k}^{3^{k+1}-1} x^{a(n)} = \sum_{i=0}^{k} \binom{k}{i} x^i 2^{k+1-i} = 2(x+2)^k.$$

Thus

$$\frac{2(x+2)^k}{3^{3k+3}} < \sum_{n=3^k}^{3^{k+1}-1} \frac{x^{a(n)}}{n^3} < \frac{2(x+2)^k}{3^{3k}},$$

and therefore

$$\frac{2}{27} \sum_{k=0}^{m} \left(\frac{x+2}{27} \right)^k < \sum_{n=1}^{3^{m+1}-1} \frac{x^{a(n)}}{n^3} < 2 \sum_{k=0}^{m} \left(\frac{x+2}{27} \right)^k.$$

It follows that the series converges (for $x > 0$) iff $(x+2)/27 < 1$; that is, $0 < x < 25$.

B-1. (148, 5, 4, 1, 0, 0, 0, 0, 1, 0, 15, 30)

Solution 1. Let I denote the value of the integral. The substitution $9 - x = y + 3$ gives

$$I = \int_2^4 \frac{\sqrt{\ln(y+3)}\, dy}{\sqrt{\ln(y+3)} + \sqrt{\ln(9-y)}},$$

so

$$2I = \int_2^4 \frac{\sqrt{\ln(x+3)} + \sqrt{\ln(9-x)}}{\sqrt{\ln(x+3)} + \sqrt{\ln(9-x)}} dx = 2, \quad \text{and} \quad I = 1.$$

Solution 2. More generally, if

$$S = \int_a^b \frac{f(x)\, dx}{f(x) + f(a+b-x)},$$

then

$$S = \int_a^b \left(1 - \frac{f(a+b-x)\, dx}{f(x) + f(a+b-x)} \right) dx$$
$$= (b-a) - \int_b^a \frac{f(t)(-dt)}{f(a+b-t) + f(t)} = (b-a) - S,$$

so $S = (b-a)/2$. In this problem $f(x) = \sqrt{\ln(9-x)}$, $a = 2$, $b = 4$, so $S = 1$.

B-2. (47, 3, 4, 0, 0, 0, 0, 0, 1, 3, 62, 84)

Solution 1. Let

$$F(r,s,t) = \frac{\binom{s}{0}}{\binom{t}{r}} + \frac{\binom{s}{1}}{\binom{t}{r+1}} + \cdots + \frac{\binom{s}{s}}{\binom{t}{r+s}}.$$

Then

$$F(r,s,t)=\frac{\binom{s-1}{0}}{\binom{t}{r}}+\frac{\binom{s-1}{0}+\binom{s-1}{1}}{\binom{t}{r+1}}$$

$$+\cdots+\frac{\binom{s-1}{s-2}+\binom{s-1}{s-1}}{\binom{t}{r+s-1}}+\frac{\binom{s-1}{s-1}}{\binom{t}{r+s}}$$

$$=F(r,s-1,t)+F(r+1,s-1,t).$$

The proof now follows easily by induction on s.

Solution 2. We find that

$$\sum_{i=0}^{s}\frac{\binom{s}{i}}{\binom{t}{r+i}}=\frac{s!r!(t-r-s)!}{t!}\sum_{i=0}^{s}\binom{r+i}{r}\binom{t-r-i}{t-r-s}.$$

Using the binomial expansion, we see that

$$\sum_{j=0}^{\infty}\binom{t-s+1+j}{t-s+1}x^j=\frac{1}{(1-x)^{t-s+2}}$$

$$=\frac{1}{(1-x)^{r+1}}\frac{1}{(1-x)^{t-r-s+1}}$$

$$=\left(\sum_{a=0}^{\infty}\binom{r+a}{r}x^a\right)\left(\sum_{b=0}^{\infty}\binom{t-r-s-b}{t-r-s}x^b\right)$$

$$=\sum_{j=0}^{\infty}\left(\sum_{i=0}^{j}\binom{r+i}{r}\binom{t-r-s+j-i}{t-r-s}\right)x^j.$$

Equating coefficients of the x^s term, we obtain

$$\binom{t+1}{t-s+1}=\sum_{i=0}^{s}\binom{r+i}{r}\binom{t-r-i}{t-r-s}.$$

Hence

$$\sum_{i=0}^{s}\frac{\binom{s}{i}}{\binom{t}{r+i}}=\frac{s!r!(t-r-s)!}{t!}\cdot\frac{(t+1)!}{s!(t-s+1)!}=\frac{t+1}{(t-s+1)\binom{t-s}{r}}.$$

Solution 3. The problem is equivalent to proving that

$$\sum_{i=0}^{s}\frac{\binom{s}{i}\binom{t-s}{r}}{\binom{t}{r+i}}=\frac{t+1}{t+1-s}.$$

It is straightforward to show that

$$\frac{\binom{s}{i}\binom{t-s}{r}}{\binom{t}{r+i}} = \frac{t+1}{t+1-s} = \frac{\binom{r+i}{r}\binom{t-r-i}{t-s-r}}{\binom{t}{s}},$$

and, therefore, letting $l = r + i$, we have

$$\sum_{i=0}^{s} \frac{\binom{s}{i}\binom{t-s}{r}}{\binom{t}{r+i}} = \sum_{i=0}^{s} \frac{\binom{r+i}{r}\binom{t-r-i}{t-s-r}}{\binom{t}{s}} = \frac{1}{\binom{t}{s}} \sum_{l=r}^{r+s} \binom{l}{r}\binom{t-l}{t-(s+r)}.$$

Now, $\sum_{l=r}^{s+r}\binom{l}{r}\binom{t-l}{t-(s+r)}$ is the number of ways to choose a sequence of t binary digits, interrupted immediately after the lth digit by a comma, such that there are r 1's preceding the comma (among the first l digits) and $t-(s+r)$ 1's following the comma (the final $t-l$ digits).

But, interpreting the comma as another symbol in the sequence, we see that this is the number of sequences of $t+1$ symbols containing s 0's, where the comma must be the $(r+1)$st nonzero symbol. Thus, the sequences are completely determined by the positions of the 0's, and so the numbers of sequences is $\binom{t+1}{s}$. Thus,

$$\sum_{l=r}^{s+r} \binom{l}{r}\binom{t-l}{t-(s+r)} = \binom{t+1}{s}$$

The result follows.

B-3. (27, 31, 5, 0, 0, 0, 0, 0, 3, 11, 57, 70)
Let

$$x_r = \frac{r^2-1}{r^2+1} \quad \text{and} \quad y_r = \frac{2r}{r^2+1}$$

for r any element of F such that $r^2 \neq -1$.

It is easy to check that $(1,0)$ and (x_r, y_r) satisfy $x^2 + y^2 = 1$.

The problem is thus to show that if x, y are in F with $(x, y) \neq (1,0)$ and $x^2 + y^2 = 1$, then $x = x_r$ and $y = y_r$ for some r. We observe that for $x_r \neq 1$, $r = y_r/(1 - x_r)$, and this suggests that we set $r = y/(1-x)$. Then we have

$$r^2 + 1 = \left(\frac{y}{1-x}\right)^2 + 1 = \frac{y^2 + (1-x)^2}{(1-x)^2}$$

$$= \frac{y^2 + x^2 - 2x + 1}{(1-x)^2} = \frac{2-2x}{(1-x)^2} = \frac{2}{1-x} \neq 0,$$

and, therefore, $r^2 - 1 = 2/(1-x) - 2 = 2x/(1-x)$. It follows that

$$x_r = \frac{r^2-1}{r^2+1} = x \quad \text{and} \quad y_r = \frac{2r}{r^2+1} = y.$$

B-4. (13, 14, 15, 1, 0, 0, 0, 0, 11, 23, 79, 48)

Let $y_0 = \operatorname{Arc}\cos 0.8$ and $\theta_n = y_0 + y_1 + \cdots + y_n$. Then $x_{n+1} = \cos\theta_n$ and $y_{n+1} = \sin\theta_n$. Since $\sin\theta = \sin(\pi - \theta) \leqslant \pi - \theta$ for $0 \leqslant \theta \leqslant \pi$, one easily proves by induction that $0 < \theta_n \leqslant \theta_{n+1} \leqslant \pi$ for $n = 1, 2, 3, \ldots$. Hence $L = \lim_{n\to\infty}\theta_n$ exists since $\theta_0, \theta_1, \ldots$ is a monotonic bounded sequence. It follows that $\lim_{n\to\infty} y_n = 0$.

Since $\cos t$ and $\sin t$ are continuous for all real t,

$$0 = \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} \sin\theta_n = \sin\left(\lim_{n\to\infty}\theta_n\right) = \sin L.$$

As $0 < L \leqslant \pi$, this implies that $L = \pi$. Then

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \cos\theta_n = \cos\left(\lim_{n\to\infty}\theta_n\right) = \cos L = \cos\pi = -1.$$

B-5. (10, 6, 3, 2, 2, 0, 0, 0, 8, 3, 16, 154)

Solution. Write $M = A + iB$ where A and B are real $2n \times n$ matrices and $N = (A, B)$, a real $2n \times 2n$ matrix. If

$$w = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = u + iv$$

where u and v are real column vectors of length n, then $\operatorname{Re}(Mw) = \operatorname{Re}[(A + iB)(u + iv)] =$

$$Au - Bv = (A \quad B)\begin{pmatrix} u \\ -v \end{pmatrix} = N\begin{pmatrix} u \\ -v \end{pmatrix}$$

and so we need to show that N is invertible. Suppose that $x = (x_1, \ldots, x_{2n})$ is a real vector of length $2n$ such that $xN = O_{2n}$. Then $xA = O_n$, $xB = O_n$ and hence $xM = x(A + iB) = O_n$. Therefore, by hypothesis, $x = O_{2n}$ and hence N has an inverse.

B-6. (8, 1, 1, 1, 0, 0, 1, 2, 1, 3, 62, 124)

Solution 1. For a in S, let $2a = \varepsilon_a s_a$ where s_a is in S and $\varepsilon_a = \pm 1$. Set $M = (p^2 - 1)/2 - N$ so that M is the number of a in S such that $\varepsilon_a = -1$. If a and b are in S and $s_a = s_b$ then $a = \pm b$ and hence $a = b$. Therefore, as a runs through S, s_a runs through S as well. Hence in F,

$$2^{(p^2-1)/2} \prod_{a\in S} a = \prod_{a\in S} (\varepsilon_a s_a) = (-1)^M \prod_{a\in S} s_a = (-1)^M \prod_{a\in S} a.$$

Hence, $(-1)^M = 2^{(p^2-1)/2}$. Using Lagrange's Theorem for finite groups or the Euler Theorem or the little Fermat Theorem, one has

$$2^{(p^2-1)/2} = \left(2^{p-1}\right)^{(p+1)/2} = (1)^{(p+1)/2} = 1$$

and, therefore, M is even. But $(p^2 - 1)/2$ is also even and therefore N is even.

Solution 2. Let Z_p denote the finite field of p elements $\{0, 1, 2, \ldots, p - 1\}$, and the elements of F by $\{a + bx\colon a, b \in Z_p\}$.

Let $H = \{1, 2, \ldots, (p - 1)/2\}$ and $S_0 = \{a + bx\colon b \in H,\ a \in Z_p$, or $b = 0$ and $a \in H\}$, and set $T_0 = 2S_0$. Then S_0 satisfies the conditions of the problem and $|S_0 \cap T_0|$ is even.

Observe that any other set S with the conditions of the problem can be obtained from S_0 by a succession of exchanges of the form "take α out of S_0 and replace it with $-\alpha$."

Suppose then that S is a set which satisfies the conditions of the problem, that $T = 2S$ and that $|S \cap T|$ is even. Suppose that $\alpha \in S$ and $S' = (S - \{\alpha\}) \cup \{-\alpha\}$, $T' = 2S'$.

If $\alpha/2 \in S$ and $2\alpha \in S$ then $|S' \cap T'| = |S \cap T| - 2$ (neither α nor 2α are in $S' \cap T'$). If $\alpha/2 \in S$ and $-2\alpha \in S$, then $|S' \cap T'| = |S \cap T|$ (α is not in $S' \cap T'$ but -2α is). If $-\alpha/2 \in S$ and $2\alpha \in S$, then $|S' \cap T'| = |S \cap T|$ (2α is not in $S' \cap T'$ but $-\alpha$ is), and if $-\alpha/2 \in S$ and $-2\alpha \in S$ then $|S' \cap T'| = |S \cap T| + 2$ (both $-\alpha$ and -2α are in $S' \cap T'$).

In each case the net change in the cardinality is 0 (mod 2), and the result follows by the preceding remarks.

Letters to the Editor

Editor:

Concerning The partial order of iterated exponentials, 93 (Dec 86) with notation as defined therein:

(1) the alleged "Theorem 6", stated without proof, is in error. For instance, in $(S_5, \prec)$, the elements [23415] and [23145] are not covers, since [32145] is in between them.

(2) The conjectures at the end of the paper fail. Their truth would strongly suggest that the partial order induced on S_n coincides with the Bruhat order on S_n. They do coincide for $n \leqslant 4$; but, for instance in S_5 again, [43215] and [34512] are comparable in the former, but not in the latter.

Professor John Stembridge has written a very interesting paper which explores the relationship between the two partial orders on S_n, and which reveals (1) and (2) in the process. I am greatly indebted to him for sharing his results with me.

Barry W. Brunson
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Editor:

It was with great interest that I read Stan Wagon's article, "Fourteen Proofs of a Result about Tiling a Rectangle," in the August/September, 1987, issue of the MONTHLY. I first heard of this result while a graduate student in a complex analysis class taught by Paul Cohen. He gave the proof using the double integral of the complex exponential function, which seemed to be a triumph of analysis over combinatorial reasoning. I have given this problem to my analysis classes sporadically over the years and, to my surprise, in the spring of 1987 a proof was found by an undergraduate, Tom Amoth. His approach, given below, appears to be different than the fourteen proofs in the article.