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The following results of the forty-ninth William Lowell Putnam Mathematical Competition, held on December 3, 1988, have been determined in accordance with the governing regulations. This annual contest is supported by the William Lowell Putnam Prize Fund for the Promotion of Scholarship, left by Mrs. Putnam in memory of her husband, and is held under the auspices of the Mathematical Association of America.

The first prize, \$5,000, was awarded to the Department of Mathematics of Harvard University. The members of the winning team were: David J. Moews, Bjorn M. Poonen, and Constantin S. Teleman; each was awarded a prize of \$250.

The second prize, \$2,500, was awarded to the Department of Mathematics of Princeton University. The members of the winning team were: Daniel J. Bernstein, David J. Grabiner, and Matthew D. Mullin; each was awarded a prize of \$200.

The third prize, \$1,500, was awarded to the Department of Mathematics of Rice University. The members of the winning team were: Hubert L. Bray, Thomas M. Hyer, and John W. McIntosh; each was awarded a prize of \$150.

The fourth prize, \$1,000, was awarded to the Department of Mathematics of the University of Waterloo. The members of the winning team were: Frank M. D'Ippolito, Colin M. Springer, and Minh-Tue Vo; each was awarded a prize of \$100.

The fifth prize, \$500, was awarded to the Department of Mathematics of the California Institute of Technology. The members of the winning team were: William P. Cross, Robert G. Southworth, and Glenn P. Tesler; each was awarded a prize of \$50.

The five highest-ranking individual contestants, in alphabetical order, were David J. Grabiner, Princeton University; Jeremy A. Kahn, Harvard University; David J. Moews, Harvard University; Bjorn M. Poonen, Harvard University; and Ravi D. Vakil, University of Toronto. Each of these students was designated a Putnam Fellow by the Mathematical Association of America and awarded a prize of \$500 by the Putnam Prize Fund.

The next six highest-ranking individuals, in alphabetical order, were William P. Cross, California Institute of Technology; Serge Elnitsky, Carleton University; Karl M. Westerberg, Carnegie-Mellon University; Glen T. Whitney, Harvard University; Sihao Wu, Yale University; and Joshua R. Zucker, Stanford University. Each was awarded a prize of \$250.

The following teams, named in alphabetical order, received honorable mention: Brown University, with team members Peter H. Golde, Kevin S. McFarland, and David J. Morin; University of California, Berkeley, with team members I-Lin Kuo, Jordan Lampe, and David P. Moulton; Carnegie-Mellon University, with team members Petros I. Hadjicostas, Joseph G. Keane, and Karl M. Westerberg;

Stanford University, with team members John C. Loftin, John A. Overdeck, and Joshua R. Zucker; and Yale University, with team members Moses G. Klein, Robert S. Manning, and William N. Nelson.

Honorable mention was achieved by the following thirty-eight individuals named in alphabetical order: Thomas R. Amoth, Oregon State University; Tibor Beke, Armand Hammer United World College; Daniel J. Bernstein, Princeton University; David T. Blackston, Massachusetts Institute of Technology; Hubert L. Bray, Rice University; Jackson A. Bross, Massachusetts Institute of Technology; Timothy K. Callahan, University of Chicago; William Chen, Washington University, St. Louis; David Cook, Harvard University; Matthew M. Cook, University of Illinois, Urbana-Champaign; Peter H. Golde, Brown University; Thomas R. Hagedorn, Princeton University; Graydon H. Hazenberg, University of Waterloo; Thomas M. Hyer, Rice University; Joseph G. Keane, Carnegie-Mellon University; Moses G. Klein, Yale University; Matthew A. Klimesh, University of Michigan, Ann Arbor; Tal N. Kubo, Harvard University; John W. McIntosh, Rice University; Christopher J. Monsour, University of Maryland, College Park; David P. Moulton, University of California, Berkeley; Matthew D. Mullin, Princeton University; Du Nguyen, University of Ottawa; John A. Overdeck, Stanford University; David L. Petry, University of Oregon; Edward R. Ratner, California Institute of Technology; Raymond M. Sidney, Harvard University; Stephen A. Smith, University of Waterloo; Robert G. Southworth, California Institute of Technology; Colin M. Springer, University of Waterloo; Constantin S. Teleman, Harvard University; John S. Tillinghast, University of California, Davis; Minh-Tue Vo, University of Waterloo; Eric K. Wepsic, Harvard University; Christopher S. Wilson, Stanford University; David Bruce Wilson, Massachusetts Institute of Technology; John H. Woo, Harvard University; and Japheth L. M. Wood, Washington University, St. Louis.

The other individuals who achieved ranks among the top 104, in alphabetical order of their schools, were: Baylor University, Adrian Tanner; University of British Columbia, Wayne J. Broughton; Brown University, David J. Morin; California Institute of Technology, Ian Agol, Earl A. Hubbell, Russell A. Manning, Glenn P. Tesler; California Polytechnic State University, Daniel L. Krejsa; University of California, Berkeley, I-Lin Kuo, Jordan Lampe; University of California, Davis, Rudolf Von Bunau; University of Chicago, Andrew S. Yeh; Cornell University, Scott S. Benson; Emory University, Charles D. McDonell; Georgia Institute of Technology, Jeffrey W. Herrmann; Harvard University, Todd A. Brun, Duff G. Campbell, Leigh Chao, Roland B. Drier, Daniel D. Lee, David M. Maymudes, Michael D. Mitzenmacher, Daniel S. Sage, Michael E. Zieve; Hofstra University, Michael Cole; Iowa State University, Brad W. Michael; Knox College, Peter F. Schultz; Massachusetts Institute of Technology, Erik Ordentlich, Deniz Yuret; Michigan State University, Steven D. Fischer, Jacob R. Lorch; University of Michigan, Ann Arbor, Robert B. Doorenbos; University of Pennsylvania, Michael Albert; Princeton University, Emory F. Bunn, Timothy Y. Chow, David C. Fox, Rahul V. Pandharipande; Reed College, Nathaniel J. Thurston; University of Rochester, Daniel B. Finn; St. Olaf College, James S. Larson; Stanford University, Thomas H. Chung, John C. Loftin; Swarthmore College, Robert E. Marx; University of Texas, Austin, Douglas S. Hauge; University of Toronto, Edward J. Doolittle; Washington University, St. Louis, Peter S. Shawhan; University of Waterloo, Paulina Chin, Frank M. D'Ippolito, Michael Glaum, Simon H. Lee, David N. C. Tse; University of Wisconsin, Madison, David G. Radcliffe; Yale University, Robert S. Manning, William M. Nelson; and Yeshiva University, Philip T. Reiss.

There were 2096 individual contestants from 360 colleges and universities in Canada and the United States in the competition of December 3, 1988. Teams were entered by 257 institutions.

The Questions Committee for the forty-ninth competition consisted of Abraham P. Hillman (Chair), Paul R. Halmos, and Gerald A. Heuer; they composed the problems listed below and were most prominent among those suggesting solutions.

PROBLEMS

Problem A-1

Let R be the region consisting of the points (x, y) of the cartesian plane satisfying both $|x| - |y| \leqslant 1$ and $|y| \leqslant 1$. Sketch the region R and find its area.

Problem A-2

A not uncommon calculus mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$. If $f(x) = e^{x^2}$, determine, with proof, whether there exists an open interval (a, b) and a nonzero function g defined on (a, b) such that this wrong product rule is true for x in (a, b).

Problem A-3

Determine, with proof, the set of real numbers x for which

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\csc\frac{1}{n} - 1\right)^{x}$$

converges.

Problem A-4

(a) If every point of the plane is painted one of three colors, do there necessarily exist two points of the same color exactly one inch apart?

(b) What if "three" is replaced by "nine"?

Justify your answers.

Problem A-5

Prove that there exists a *unique* function f from the set $\mathbf{R}^+$ of positive real numbers to $\mathbf{R}^+$ such that

$$f(f(x)) = 6x - f(x) \quad \text{and} \quad f(x) > 0 \quad \text{for all } x > 0.$$

Problem A-6

If a linear transformation $\mathbf{A}$ on an n-dimensional vector space has $n + 1$ eigenvectors such that any n of them are linearly independent, does it follow that $\mathbf{A}$ is a scalar multiple of the identity? Prove your answer.

Problem B-1

A *composite* (positive integer) is a product ab with a and b not necessarily distinct integers in $\{2, 3, 4, \dots\}$. Show that every composite is expressible as $xy + xz + yz + 1$, with x, y, and z positive integers.

Problem B-2

Prove or disprove: If x and y are real numbers with $y \geqslant 0$ and $y(y+1) \leqslant (x+1)^2$, then $y(y-1) \leqslant x^2$.

Problem B-3

For every n in the set $\mathbf{Z}^+ = \{1, 2, \dots\}$ of positive integers, let r_n be the minimum value of $|c - d\sqrt{3}|$ for all nonnegative integers c and d with $c + d = n$. Find, with proof, the smallest positive real number g with $r_n \leqslant g$ for all n in $\mathbf{Z}^+$.

Problem B-4

Prove that if $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive real numbers, then so is $\sum_{n=1}^{\infty} (a_n)^{n/(n+1)}$.

Problem B-5

For positive integers n, let $\mathbf{M}_n$ be the $2n+1$ by $2n+1$ skew-symmetric matrix for which each entry in the first n subdiagonals below the main diagonal is 1 and each of the remaining entries below the main diagonal is -1. Find, with proof, the rank of $\mathbf{M}_n$. (According to one definition the rank of a matrix is the largest k such that there is a $k \times k$ submatrix with nonzero determinant.)

One may note that

$$\mathbf{M}_1 = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \text{ and } \mathbf{M}_2 = \begin{pmatrix} 0 & -1 & -1 & 1 & 1 \\ 1 & 0 & -1 & -1 & 1 \\ 1 & 1 & 0 & -1 & -1 \\ -1 & 1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 1 & 0 \end{pmatrix}.$$

Problem B-6

Prove that there exist an infinite number of ordered pairs (a, b) of integers such that for every positive integer t the number $at + b$ is a triangular number if and only if t is a triangular number.

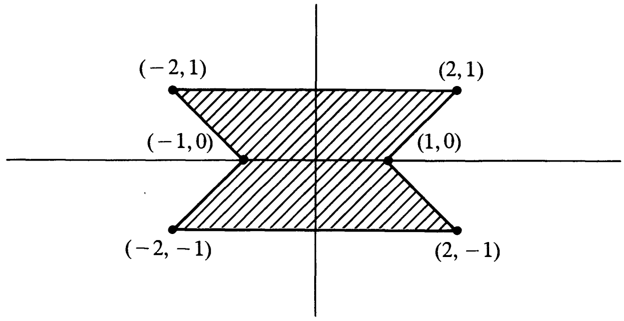
(The triangular numbers are the $t_n = n(n+1)/2$ with n in $\{0, 1, 2, \dots\}$.)

SOLUTIONS

In the 12-tuples $(n_{10}, n_9, \dots, n_0, n_{-1})$ following each problem number below, n_i for $10 \geqslant i \geqslant 0$ is the number of students among the top 208 contestants achieving i points for the problem and n_{-1} is the number of those not submitting solutions.

A-1. (202, 0, 0, 0, 0, 0, 0, 0, 4, 2, 0, 0)

The part of R in the first quadrant is bounded by $x = 0$, $y = 0$, $x - y = 1$, and $y = 1$. This part is a trapezoid with vertices $(0,0)$, $(1,0)$, $(2,1)$, and $(0,1)$ and area $3/2$. Since $(\pm x, \pm y)$ is in R when (x, y) is in R, the parts of R in the other quadrants are obtained using symmetry about both axes, and consequently, the area of R is 6.



A-2. (174, 1, 7, 1, 0, 0, 0, 3, 3, 0, 15, 4)

The function defined by $g(x) = e^x\sqrt{2x-1}$ has the property desired for $1/2 < a < x < b$ and $g(x) = e^x\sqrt{1-2x}$ has the property for $a < x < b < 1/2$.

To derive that result, consider the equation $(fg)' = f'g'$ and rewrite it in the successive forms

$$f'(x)g(x) + f(x)g'(x) = f'(x)g'(x),$$

$$\frac{g'(x)}{g(x)} = \frac{-f'(x)/f(x)}{1 - f'(x)/f(x)}.$$

If $f(x) = e^{x^2}$ then we have

$$\frac{g'(x)}{g(x)} = \frac{-2x}{1-2x}$$

$$\log|g(x)| = x + \tfrac{1}{2}\log|1-2x| + C,$$

where C is an arbitrary constant. If $1/2 < a < x < b$, this has the form

$$g(x) = Ae^x\sqrt{2x-1},$$

where A is an arbitrary positive real number. If $a < x < b < 1/2$, it has the form $g(x) = Ae^x\sqrt{1-2x}$.

A-3. (42, 17, 10, 0, 0, 0, 0, 6, 5, 3, 34, 91)

Let

$$a_n = \frac{1}{n}\csc\frac{1}{n} - 1.$$

Then

$$a_n = \frac{1}{n\sin\frac{1}{n}} - 1 = \frac{1}{n\left(\frac{1}{n} - \frac{1}{6n^3} + \frac{1}{120n^5} - \cdots\right)} - 1$$

$$= \frac{1}{1 - \frac{1}{6n^2} + \frac{1}{120n^4} + \cdots} - 1$$

$$= 1 + \frac{1}{6n^2} + \frac{1}{n^2}g(n) - 1 = \frac{1}{n^2}\left(\frac{1}{6} + g(n)\right),$$

where $g(n) \to 0$ as $n \to \infty$. Hence, there exist positive real numbers c, d, and N such that

$$c\frac{1}{n^2} \leqslant a_n \leqslant d\frac{1}{n^2}, \quad \text{for } n > N.$$

Using the comparison and the p-test, one finds that Σa_n^x converges for $x > 1/2$ and diverges for $0 < x \leqslant 1/2$. But it is easy to see that the series also diverges for $x \leqslant 0$. Hence the answer is $\{x : x > 1/2\}$.

A-4. (45, 29, 2, 2, 1, 25, 11, 0, 3, 3, 25, 62)

(a) The answer is yes. For the proof let A be an arbitrary point in the plane and let ABC be an arbitrary equilateral triangle with side length 1 (where the units are inches, of course) that has A as one of its vertices. If any two of A, B, and C have the same color, the construction is finished. If not, let A' be the point obtained by reflecting A through the line BC. If A' has the same color as either B or C, the construction is finished. If not, then A and A' have the same color. Note that the distance between A and A' is $\sqrt{3}$, and that, in fact, any two points at distance $\sqrt{3}$ from one another can be obtained by making one of them a vertex of an equilateral triangle of side length 1 and then reflecting it through the side opposite it.

The result so obtained implies that, for any initial point A, either the reflected equilateral triangle argument finishes the desired construction for some B and C, or else that every point at distance $\sqrt{3}$ from A has the same color as A. The set of points such as A', at distance $\sqrt{3}$ from A, is a circle of radius $\sqrt{3}$; any chord of length 1, of that circle, yields a pair of points of the same color exactly one inch apart.

(b) The answer is no. For the proof, pave the plane with squares whose common side length is chosen so that the diagonals are nearly 1 but not equal to 1; the diagonal length 0.9 will do. If that length is used, then the side length of each square is $0.9/\sqrt{2}$, which is somewhat greater than 0.63. Color one square with color #1, color the eight squares adjacent to it with colors #2–#9, and then repeat, throughout the plane, the coloring scheme of the large square (consisting of nine small squares) so obtained. (For present purposes it doesn't matter what consistent convention is followed for the boundaries of the squares; one possibility is to let the bottom and left boundaries of each square have the same color as the interior.)

The result is a nine-coloring of the plane in which no two points of the same color are exactly one inch apart. Indeed, for any point at all, the points of the same color are either within 0.9 inches from it or else farther than $2 \times .63 = 1.26$ inches.

A-5. (32, 8, 4, 0, 0, 0, 0, 1, 7, 96, 9, 51)

For arbitrary $x > 0$, let $a_0, a_1, a_2, \ldots$ be defined by $a_0 = x$ and $a_{n+1} = f(a_n)$. Then $a_{n+2} + a_{n+1} - 6a_n = 0$ for $n = 0, 1, 2, \ldots$. The characteristic roots of this difference equation are -3 and 2. Hence $a_n = (-3)^n c + 2^n k$ for some constants c and k. As $a_{n+1} = f(a_n) > 0$ for all n, we must have $c = 0$ and so $f(x) = 2x$. This unique f satisfies the conditions since it gives $f(f(x)) = f(2x) = 4x = 6x - f(x)$ and $2x > 0$ for $x > 0$.

A-6. (59, 14, 10, 7, 0, 0, 0, 0, 5, 0, 21, 92)

Yes, $\mathbf{A}$ must be a scalar multiple of the identity. Suppose that $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{n+1}$ are eigenvectors of $\mathbf{A}$ such that any n of them are linearly independent, with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{n+1}$. Let $B_i = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{n+1}\} \setminus \{\mathbf{x}_i\}$. Then B_i is a set of n linearly independent vectors in an n-dimensional vector space, so B_i is a basis. With respect to B_i, the transformation $\mathbf{A}$ is represented by a diagonal matrix, $\operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_n)$. Thus, trace($\mathbf{A}$) $= S - \lambda_i$, where S is the sum of the eigenvalues, $S = \lambda_1 + \lambda_2 + \cdots + \lambda_{n+1}$. But the trace of a linear transformation is independent of the basis chosen. Thus,

$$S - \lambda_i = S - \lambda_j \quad \text{for all } i, j$$

$$\lambda_i = \lambda_j \quad \text{for all } i, j$$

Let λ be this common value. Then with respect to any of the bases B_i, $\mathbf{A}$ corresponds to $\operatorname{diag}(\lambda, \lambda, \ldots, \lambda)$, which is λ times the identity.

B-1. (176, 25, 0, 0, 0, 0, 0, 0, 1, 1, 1, 4)
Letting $z = 1$, we have

$$xy + xz + yz + 1 = xy + x + y + 1 = (x+1)(y+1)$$

and this gives us all the composite positive integers when x and y range over all the positive integers.

B-2. (88, 38, 30, 0, 0, 0, 0, 0, 26, 4, 18, 4)
The desired conclusion is true for $0 \leqslant y \leqslant 1$, so suppose $y > 1$. If $(x+1)^2 \geqslant y(y+1)$ then we may assume that $x > 0$, and therefore $x \geqslant \sqrt{y(y+1)} - 1 \geqslant \sqrt{y(y-1)}$, and the result follows. (The last inequality follows from the easily verified fact that for positive numbers a and b, $\sqrt{ab} + 1 \leqslant \sqrt{(a+1)(b+1)}$.)

B-3. (20, 16, 17, 2, 0, 0, 0, 0, 52, 33, 27, 41)
Let $g = (1 + \sqrt{3})/2$. Since for each fixed value of n the sequence $n, n - 1 - \sqrt{3}, n - 2 - 2\sqrt{3}, \ldots, -n\sqrt{3}$ is an arithmetic progression with $-2g$ as common difference, there is a unique term x_n in it with $-g < x_n < g$. Clearly $r_n = |x_n|$. Let $\varepsilon > 0$. By the pigeonhole principle, there exist a and b with $a \neq b$ and $|x_a - x_b| < \varepsilon$. Let $t = |a - b|$. In the sequence $r_t, r_{2t}, r_{3t}, \ldots$ there is an r_{kt} such that $g - \varepsilon < r_{kt} \leqslant g$. Hence g is the desired least upper bound of the r_n.

B-4. (17, 1, 0, 0, 0, 0, 0, 0, 3, 2, 73, 112)
Let $S = \{n : a_n^{n/n+1} < 2a_n\}$. If $n \notin S$, $a_n^{n/n+1} \geqslant 2a_n$, or equivalently $1/2 \geqslant a_n^{1-(n/n+1)} = a_n^{1/n+1}$, which is the same as $1/2^n \geqslant a_n^{n/n+1}$. It follows that

$$\sum_{n=1}^{\infty} a_n^{n/n+1} \leqslant \sum_{n \in S} a_n^{n/n+1} + \sum_{n \notin S} 1/2^n < \infty.$$

B-5. (9, 0, 10, 1, 0, 0, 0, 0, 0, 1, 2, 45, 140)
Let u be a primitive kth root of 1, where $k = 2n + 1$. For $1 \leqslant i \leqslant k$, let $\mathbf{L}_i$ denote the column vector $(1, u^{i-1}, u^{2(i-1)}, \ldots, u^{(k-1)(i-1)})$. The k by k matrix whose ith column is $\mathbf{L}_i$ is a Vandermonde matrix, so the $\mathbf{L}_i$ are linearly independent over the complex numbers. For $1 \leqslant i \leqslant k$ we have $\mathbf{M}_n\mathbf{L}_i = c_i\mathbf{L}_i$, where c_i is the scalar dot product of $\mathbf{L}_i$ with the first row of $\mathbf{M}_n$. Note that $c_i = 0$, but for $2 \leqslant i \leqslant k$, $c_i = -\sum_{j=2}^{n+1} u^{(j-1)(i-1)} + \sum_{j=n+2}^{k} u^{(j-1)(i-1)} \neq 0$. It follows that the vectors $\{c_2\mathbf{L}_2, \ldots, c_k\mathbf{L}_k\}$, and hence the vectors $\{\mathbf{M}_n\mathbf{L}_2, \ldots, \mathbf{M}_n\mathbf{L}_k\}$, are linearly independent. But $\mathbf{M}_n\mathbf{L}_1 = c_1\mathbf{L}_1 = 0$, so $\mathbf{M}_n$ has rank $2n$.

B-6. (38, 9, 4, 4, 24, 8, 0, 5, 5, 1, 17, 93)

Solution 1. It is easy to see that $t_{3n+1} = 1 + 9t_n$ and that $t_{3n} \equiv t_{3n+2} \equiv 0 \pmod 3$. This implies that (9, 1) is one of our ordered pairs. If the numbers a_m and b_m are defined by

$$f(x) = 9x + 1, \qquad f(9x + 1) = a_2x + b_2, \ldots,$$
$$f(a_m x + b_m) = a_{m+1}x + b_{m+1},$$

an easy induction on m shows that (a_m, b_m) has the desired properties for $m = 2, 3, \ldots$.

Solution 2. We show that the ordered pairs $(8t_r + 1, t_r)$ have the desired properties. Let $T = \{0, 1, 3, 6, \ldots\}$ be the set of triangular numbers and $Q = \{1, 9, 25, 49, \ldots\}$ be the set of squares of odd integers. The equality $(2n + 1)^2 = 8((n^2 + n)/2) + 1$ implies that

$$t \text{ is in } T \text{ if and only if } 8t + 1 \text{ is in } Q. \qquad (*)$$

Let $t_r = r(r + 1)/2$ be in T and $q = 8t_r + 1$.

For the "if" part, let t be in T. Since Q is closed under multiplication and $8t + 1$ is in Q by $(*)$, we see that

$$q(8t + 1) = 8qt + q = 8qt + 8t_r + 1 = 8(qt + t_r) + 1$$

is in Q and hence $qt + t_r$ is in T by $(*)$. This proves the "if" part.

For the "only if" part, let t be an integer and $qt + t_r$ be in T. Then

$$8(qt + t_r) + 1 = 8[(8t_r + 1)t + t_r] + 1 = (8t_r + 1)(8t + 1)$$

is in Q. Since $8t + 1$ is an integer and is the quotient of squares in Q, it follows that $8t + 1$ itself is in Q. Then $(*)$ tells us that t is in T. This completes the proof.