The Fiftieth William Lowell Putnam Mathematical Competition

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The following results of the fiftieth William Lowell Putnam Mathematical Competition, held on December 2, 1989, have been determined in accordance with the governing regulations. This annual contest is supported by the William Lowell Putnam Prize Fund for the Promotion of Scholarship, left by Mrs. Putnam in memory of her husband, and is held under the auspices of the Mathematical Association of America.

The first prize, \$5,000, was awarded to the Department of Mathematics of Harvard University. The members of the winning team were: Jeremy A. Kahn, Raymond M. Sidney, and Eric K. Wepsic; each was awarded a prize of \$250.

The second prize, \$2,500, was awarded to the Department of Mathematics of Princeton University. The members of the winning team were: David J. Grabiner, Mathew D. Mullin, and Rahul V. Pandharipande; each was awarded a prize of \$200.

The third prize, \$1,500, was awarded to the Department of Mathematics of the University of Waterloo. The members of the winning team were: Grayden Hazenberg, Stephen M. Smith, and Colin M. Springer; each was awarded a prize of \$150.

The fourth prize, \$1,000, was awarded to the Department of Mathematics of Yale University. The members of the winning team were: Bruce E. Kaskel, Andrew H. Kresch, and Sihao Wu; each was awarded a prize of \$100.

The fifth prize, \$500, was awarded to the Department of Mathematics of Rice University. The members of the winning team were: Hubert L. Bray, John W. McIntosh, and David S. Metzler; each was awarded a prize of \$50.

The six highest ranking individual contestants, in alphabetical order, were Christo Athanasiadis, Massachusetts Institute of Technology; William P. Cross, California Institute of Technology; Andrew H. Kresch, Yale University; Colin M. Springer, University of Waterloo; Ravi D. Vakil, University of Toronto; and Sihao Wu, Yale University. Each of these was designated a Putnam Fellow by the Mathematical Association of America and awarded a prize of \$500 by the Putnam Prize Fund.

The next four highest ranking individuals, in alphabetical order, were William Chen, Washington University, St. Louis; Jordan S. Ellenberg, Harvard University; David J. Grabiner, Princeton University; and Raymond M. Sidney, Harvard University. Each was awarded a prize of \$250.

The following teams, named in alphabetical order, received honorable mention: California Institute of Technology, with team members William P. Cross, Robert Southworth, and Glenn Tesler; the University of California, Berkeley, with team members Warren M. Lam, Jordan Lampe, and Jonathan C. Ruyle; Carnegie Mellon University, with team members Petros I. Hadjicostas, Eric Lauer, and Karl M. Westerberg; the University of Chicago, with team members Justin Boyan, Doyle Tanner, and Andrew Yeh; and Stanford University, with team members John C. Loftin, Greg G. Martin, and Christopher S. Wilson.

Honorable mention was achieved by the following thirty-five individuals named in alphabetical order: Edgar D. Bailey, Jr., University of Texas, Austin; Daniel J. Bernstein, New York University; Hubert L. Bray, Rice University; Nicholai I. Chaudarov, Brandeis University; Andrew Chou, Massachusetts Institute of Technology; Michael Cole, Hofstra University; Matthew M. Cook, University of Illinois, Champaign-Urbana; Daniel P. Cory, Stanford University; Kevin B. Ford, California State University, Chico; Michael L. Hutchings, Harvard University; Jeremy A. Kahn, Harvard University; Jordan Lampe, University of California, Berkeley; Gregory D. Landweber, Princeton University; Daniel D. Lee, Harvard University; Daniel I. Lieberman, Princeton University; Sanjoy S. Mahajan, Stanford University; Russell A. Manning, California Institute of Technology; David M. Maymudes, Harvard University; John W. McIntosh, Rice University; Christopher J. Monsour, University of Maryland, College Park; Rahul Vijay Pandharipande, Princeton University; Eric M. Rains, Case Western Reserve University; Stephen A. Smith, University of Waterloo; Steven M. Stadnicki, Clarkson University; Andras Vasy, Stanford University; Eric H. Veach, University of Waterloo; Nikolai I. Weaver, Harvey Mudd College; Eric K. Wepsic, Harvard University; Karl M. Westerberg, Carnegie Mellon University; David B. Wilson, Massachusetts Institute of Technology; Michael P. Wolf, Harvard University; Ali F. Yegulalp, Columbia University; Andrew Yeh, University of Chicago; Zhaoliang Zhu, Yale University; and Michael E. Zieve, Harvard University.

The other individuals who achieved ranks among the top 94, in alphabetical order of their schools, were: University of Alberta, Graham C. Denham, Brendan M. Mumey; University of British Columbia, Wayne J. Broughton, Gregory F. Wellman; Brown University, Martin M. Wattenberg; California Institute of Technology, Tien-Yee Chiu, Michael G. Greenblatt, Gwoho H. Liu, Robert G. Southworth; University of California, Berkeley, Bryan F. Clair, Wee-Lang Heng; University of California, Santa Barbara, John C. Carey; University of California, San Diego, Franz J. Wrasidlo; University of Chicago, Adrian Tanner; Duke University, Will A. Schneeberger; Emory University, Charles D. McDonell; Georgia Institute of Technology, Jeffrey W. Herrmann; Harvard University, J. Stewart Burns, Roland B. Dreier, Daniel Eric Gottesman, F. Dean Hildebrandt, Michael C. Jablecki, Michael D. Mitzenmacher, Matthew C. Weiner; University of Maryland, College Park, Eric M. Boesch, Jay P. Chawla; Massachusetts Institute of Technology, Todd W. Rowland; Michigan State University, Jacob R. Lorch; Princeton University, Timothy Y. Chow, Paul E. Ericksen, William P. Minicozzi, Matthew D. Mullin; Reed College, Nathaniel P. Thurston; Rice University, John Kenneth Burton, Jr., David S. Metzler; Simon Fraser University, Richard S. Kiss; Stanford University, Thomas H. Chung, John C. Loftin, Greg G. Martin; University of Texas, Austin, Bryan W. Taylor; Trinity College, Hartford, Marshall A. Whittlesey; Washington University, St. Louis, Adam M. Costello, Jordan A. Samuels, Peter S. Shawhan; University of Waterloo, Graydon H. Hazenberg, Marc S. Ordower, Giuseppe Russo; Wellesley College, Yihao L. Zhang; and the University of Wisconsin, Madison, Cavan C. Fang.

There were 2392 individual contestants from 373 colleges and universities in Canada and the United States in the competition of December 2, 1989. Teams were entered by 288 institutions.

The Questions Committee for the fiftieth competition consisted of Gerald A. Heuer (Chair), Paul R. Halmos, and Kenneth A. Stolarsky; they composed the problems listed below and were most prominent among those suggesting solutions.

PROBLEMS

Problem A-1. How many primes among the positive integers, written as usual in base 10, are such that their digits are alternating 1's and 0's, beginning and ending with 1?

Problem A-2. Evaluate $\int_0^a \int_0^b e^{\max\{b^2x^2,\, a^2y^2\}}\, dy\, dx$, where a and b are positive.

Problem A-3. Prove that if

$$11z^{10} + 10iz^9 + 10iz - 11 = 0,$$

then $|z| = 1$. (Here z is a complex number and $i^2 = -1$.)

Problem A-4. If α is an irrational number, $0 < \alpha < 1$, is there a finite game with an honest coin such that the probability of one player winning the game is α? (An honest coin is one for which the probability of heads and the probability of tails are both 1/2. A game is finite if with probability 1 it must end in a finite number of moves.)

Problem A-5. Let m be a positive integer and let $\mathscr{G}$ be a regular $(2m + 1)$-gon inscribed in the unit circle. Show that there is a positive constant A, independent of m, with the following property. For any point p inside $\mathscr{G}$ there are two distinct vertices v_1 and v_2 of $\mathscr{G}$ such that

$$\big|\, |p - v_1| - |p - v_2| \,\big| < \frac{1}{m} - \frac{A}{m^3}.$$

Here $|s - t|$ denotes the distance between the points s and t.

Problem A-6. Let $\alpha = 1 + a_1x + a_2x^2 + \cdots$ be a formal power series with coefficients in the field of two elements. Let

$$a_n = \begin{cases} 1 & \text{if every block of zeros in the binary expansion of } n \text{ has an even number of zeros in the block,} \\ 0 & \text{otherwise.} \end{cases}$$

(For example, $a_{36} = 1$ because $36 = 100100_2$, and $a_{20} = 0$ because $20 = 10100_2$.) Prove that $\alpha^3 + x\alpha + 1 = 0$.

Problem B-1. A dart, thrown at random, hits a square target. Assuming that any two parts of the target of equal area are equally likely to be hit, find the probability that the point hit is nearer to the center than to any edge. Express your answer in the form $(a\sqrt{b} + c)/d$, where a, b, c, d are positive integers.

Problem B-2. Let S be non-empty set with an associative operation that is left and right cancellative ($xy = xz$ implies $y = z$, and $yx = zx$ implies $y = z$). Assume that for every a in S the set $\{a^n\colon n = 1, 2, 3, \ldots\}$ is finite. Must S be a group?

Problem B-3. Let f be a function on $[0,\infty)$, differentiable and satisfying

$$f'(x) = -3f(x) + 6f(2x)$$

for $x > 0$. Assume that $|f(x)| \leqslant e^{-\sqrt{x}}$ for $x \geqslant 0$ (so that $f(x)$ tends rapidly to 0 as x increases). For n a non-negative integer, define

$$\mu_n = \int_0^\infty x^n f(x)\,dx$$

(sometimes called the nth moment of f).

a. Express μ_n in terms of μ_0.

b. Prove that the sequence $\{\mu_n(3^n/n!)\}$ always converges, and that the limit is 0 only if $\mu_0 = 0$.

Problem B-4. Can a countably infinite set have an uncountable collection of non-empty subsets such that the intersection of any two of them is finite?

Problem B-5. Label the vertices of a trapezoid T (quadrilateral with two parallel sides) inscribed in the unit circle as A, B, C, D so that AB is parallel to CD and A, B, C, D are in counterclockwise order. Let s_1, s_2 and d denote the lengths of the line segments AB, CD, and OE, where E is the point of intersection of the diagonals of T, and O is the center of the circle. Determine the least upper bound of $(s_1 - s_2)/d$ over all such T for which $d \neq 0$, and describe all cases, if any, in which it is attained.

Problem B-6. Let $(x_1, x_2, \ldots, x_n)$ be a point chosen at random from the n-dimensional region defined by $0 < x_1 < x_2 < \cdots < x_n < 1$. Let f be a continuous function on $[0,1]$ with $f(1) = 0$. Set $x_0 = 0$ and $x_{n+1} = 1$. Show that the expected value of the Riemann sum

$$\sum_{i=0}^{n} (x_{i+1} - x_i) f(x_{i+1})$$

is $\int_0^1 f(t)P(t)\,dt$, where P is a polynomial of degree n, independent of f, with $0 \leqslant P(t) \leqslant 1$ for $0 \leqslant t \leqslant 1$.

SOLUTIONS

In the 12-tuples $(n_{10}, n_9, \ldots, n_0, n_{-1})$ following each problem number below, n_i for $10 \geqslant i \geqslant 0$ is the number of students among the top 199 contestants achieving i points for the problem and n_{-1} is the number of those not submitting solutions.

A-1 (92, 2, 6, 7, 0, 0, 0, 9, 2, 8, 41, 32)

Solution. One. Let N_k be the number

$$101010\ldots101$$

with exactly k digits equal to 0. If $k = 1$, so that $N_k = 101$, then N_k is prime. All other N_k's are composite, as the following reasoning shows. If k is odd, then 101 divides N_k. If k is even, then

$$11N_k = R \cdot S,$$

where R is the number consisting of $k + 1$ digits all equal to 1 and S is the number with $k+2$ digits beginning and ending with 1 and having only 0's in between. One of the numbers R and S divides N_k.

A-2 (141, 6, 29, 0, 0, 0, 0, 0, 5, 7, 4, 7)

Solution. Divide the region into two parts by the diagonal line $ay = bx$ to get

$$\int_0^a \int_0^b e^{\max\{b^2x^2, a^2y^2\}}\, dy\, dx = \int_0^a \int_0^{bx/a} e^{b^2x^2}\, dy\, dx + \int_0^b \int_0^{ay/b} e^{a^2y^2}\, dx\, dy$$
$$= \int_0^a \frac{bx}{a} e^{b^2x^2}\, dx + \int_0^b \frac{ay}{b} e^{a^2y^2}\, dy$$
$$= \frac{e^{a^2b^2} - 1}{ab}.$$

A-3 (13, 4, 4, 2, 0, 0, 0, 0, 6, 8, 56, 106)

Solution. We have $z^9 = \dfrac{11 - 10iz}{11z + 10i}$. If $z = a + bi$, then

$$|z^9| = \left|\frac{11 - 10iz}{11z + 10i}\right| = \sqrt{\frac{|11^2 + 220b + 10^2(a^2+b^2)|}{|11^2(a^2+b^2) + 220b + 10^2|}} \equiv \frac{f(a,b)}{g(a,b)}.$$

If $a^2 + b^2 > 1$, then $g(a,b) > f(a,b)$, making $|z^9| < 1$, a contradiction. If $a^2 + b^2 < 1$, then $f(a,b) > g(a,b)$, making $|z^9| > 1$, a contradiction. Thus, $|z| = 1$.

Second Solution. Rouché's Theorem states that if $|g(z) - f(z)| < |f(z)|$ for z on $|z| = \alpha$ then g and f have the same number of zeros inside $|z| = \alpha$. Let $g(z) = 11z^{10} + 10iz^9 + 10iz - 11$ and $f(z) = 10iz - 11$. Then $|g(z) - f(z)| = |11z^{10} + 10iz^9| = |z|^9|i|\,|11z + 10i| < |10iz - 11|$ if $|z| < 1$. Now $f(z)$ has its only root at $11/10i$ so that $g(z)$ has no zeros inside $|z| < 1$. If z is a zero of g and $|z| > 1$, then $-1/z$ is also a zero of g (which we've seen doesn't happen), and therefore, the only zeros of g are on the circle $|z| = 1$.

A-4 (54, 22, 16, 4, 0, 0, 0, 0, 4, 2, 53, 44)

Solution. Yes. Write the number

$$\alpha = \sum_{n=1}^{\infty} \frac{\alpha_n}{2^n}$$

in its binary representation,

$$\alpha_n = 0 \text{ or } 1, \qquad n = 1, 2, 3, \ldots,$$

and play the game as follows. Keep tossing the coin, recording the result, say β_n, of the nth toss as 0 when it falls tails and 1 when it falls heads; you win the game if the first time that $\beta_n \neq \alpha_n$ your result, β_n, is 0 (and, therefore, $\alpha_n = 1$). In other words, you win if at the first time when the β sequence differs from the α sequence, the β sequence is smaller.

The motivation for this procedure is to think of α as determining the interval $(0, \alpha)$, and think of the coin tosses as determining a number in $[0, 1]$; "win" means "land in the prescribed interval". There are many other solutions; this might be one of the simplest.

How can you win? To see the answer, it helps to introduce some notation: let $n_1, n_2, n_3, \ldots$ be the positions of the 1's that occur in the sequence, α. One way to win is to toss so that the β sequence agrees with the α sequence for the first $n_1 - 1$ terms and then disagrees. The probability of that is $1/2$ to the power n_1, which is equal to the partial sum up to n_1 of the binary representation of α. The next way to win is to toss so that the β sequence agrees with the α sequence for the first $n_2 - 1$ terms and then disagrees. The probability of that is $1/2$ to the n_2, which is equal to the number obtained from the partial sum up to n_2 of the binary representation of α by replacing the first 1 with a 0. These two ways of winning are the beginning of an infinite sequence. The (infinite) sum of the corresponding probabilities is exactly the binary representation of α.

A-5 (6, 1, 3, 0, 0, 0, 0, 1, 0, 2, 49, 137)

Solution. The diameter of $\mathscr{G}$ is $2\cos(\pi/2(2m+1))$, so all the $|p - v_i|$ for $1 \leqslant i \leqslant 2m+1$ fall into the interval $[0, 2\cos(\pi/2(2m+1))]$. Hence some two of them differ by at most

$$\frac{2\cos\left(\dfrac{\pi}{2(2m+1)}\right)}{2m}.$$

Let

$$f(m) = \frac{1 - \cos\left(\dfrac{\pi}{2(2m+1)}\right)}{\left(\dfrac{1}{m}\right)^2}.$$

This is a positive term sequence with limit $\pi^2/32$ and therefore this sequence has a positive minimum value. Therefore,

$$\frac{1 - \cos\left(\dfrac{\pi}{2(2m+1)}\right)}{\left(\dfrac{1}{m}\right)^2} > A$$

for some $A > 0$. Hence

$$\cos\left(\frac{\pi}{2(2m+1)}\right) < 1 - \frac{A}{m^2}$$

and the result follows.

A-6 (3, 1, 0, 0, 0, 0, 0, 1, 0, 4, 49, 141)

Solution. We show that $\alpha^4 + x\alpha^2 + \alpha = 0$.

Note that

$$\alpha^2 = \left(\sum_{n=0}^{\infty} a_n x^n\right)^2 = \sum_{n=0}^{\infty} a_n x^{2n}, \qquad \alpha^4 = \sum_{n=0}^{\infty} a_n x^{4n}, \qquad x\alpha^2 = \sum_{n=0}^{\infty} a_n x^{2n+1}.$$

Therefore,

$$\alpha^4 + x\alpha^2 + \alpha = \sum_{\substack{n=0\\ 2|n,\, 4|n}}^{\infty} a_n x^n + \sum_{\substack{n=0\\ 4|n}}^{\infty} (a_{n/4} + a_n) x^n + \sum_{\substack{n=0\\ n \text{ odd}}}^{\infty} (a_n + a_{(n-1)/2}) x^n.$$

If $2|n, 4|n$, then $a_n = 0$. If $4|n$ then $a_{n/4} = a_n$, so $a_{n/4} + a_n = 0$. If n is odd, then $a_{(n-1)/2} = a_n$, so $a_n + a_{(n-1)/2} = 0$.

B-1 (144, 10, 0, 0, 0, 0, 0, 0, 0, 0, 42, 3)

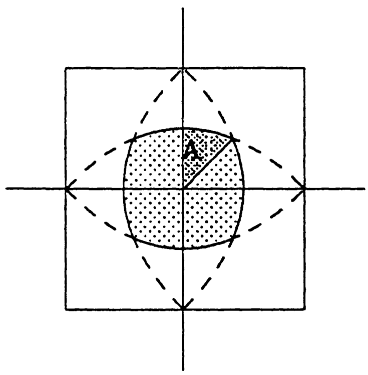
Solution. Consider a 2×2 dartboard centered at the origin with vertices at $(1,1)$, $(-1,1)$, $(-1,-1)$, and $(1,-1)$. A point (x, y) in the square is closer to the center of the board than the top edge of the board if and only if

$$\sqrt{x^2 + y^2} \leqslant 1 - y,$$

or equivalently,

$$y \leqslant \frac{1 - x^2}{2}.$$

A similar parabolic region arises for each of the other three sides, and therefore the region of points closer to the center than any edge has the shape of the shaded region in the following figure.



Let A denote the area of this region in the first quadrant bounded by $y = (1 - x^2)/2$ and $y = x$. Then, the probability we desire is

$$\frac{8 \text{ Area } A}{4} = 2 \text{ Area } A = 2\int_0^{\sqrt{2}-1}\left(\frac{1 - x^2}{2} - x\right) dx = \frac{4\sqrt{2} - 5}{3}.$$

B-2 (49, 44, 17, 10, 0, 0, 0, 0, 3, 4, 41, 31)

Solution. Yes. Let a be an arbitrary element of S. The set $\{a^n \colon n = 1, 2, 3, \dots\}$ is finite, and therefore $a^m = a^n$ for some m, n with $m > n \geqslant 1$. By cancellation we have $a^{r(a)} = a$, where $r(a) = m - n + 1 > 1$. If x is any element of S, then $aa^{r(a)-1}x = a^{r(a)}x = ax$, and this implies that $a^{r(a)-1}x = x$. Similarly, we see that $xa^{r(a)-1} = x$, and the element $e \equiv a^{r(a)-1}$ is an identity. The identity element is unique, for if e' is another identity, then $e = ee' = e'$. If $r(a) > 2$ then $a^{r(a)-2}$ is an inverse of a, and if $r(a) = 2$ then $a^2 = a = e$ and a is its own inverse. Thus S is a group.

B-3 (36, 4, 18, 48, 4, 10, 3, 1, 4, 1, 9, 61)

Solution. a. Clearly, $\int_0^\infty x^n g(x)\,dx$ exists for $g(x)$ equal to $f(x)$ or $f'(x)$ because $f(x)$ tends rapidly to 0. Hence,

$$\int_0^\infty x^n f'(x)\,dx = -3\mu_n + 6\int_0^\infty x^n f(2x)\,dx.$$

Using parts on the integral on the left and the substitution $u = 2x$ on the integral on the right, we obtain

$$x^n f(x)]_0^\infty - n\mu_{n-1} = -3\mu_n + \frac{6}{2^{n+1}}\mu_n.$$

Since $x^n f(x) \to 0$ as $x \to \infty$ for any $n > 0$ we have

$$\mu_n = \frac{n}{3}\frac{1}{1-\dfrac{1}{2^n}}\mu_{n-1}.$$

Iteration now yields

$$\mu_n = \frac{n!}{3^n}\frac{1}{\prod_{m=1}^{n}\left(1-\dfrac{1}{2^m}\right)}\mu_0.$$

b. Since $\Sigma_{m=1}^\infty(1/2^m) < \infty$, the infinite product $\Pi_{m=1}^\infty(1-(1/2^m))$ converges to a nonzero finite limit and the result follows.

B-4 (52, 7, 0, 0, 0, 0, 0, 0, 0, 1, 62, 77)

Solution. Yes. Consider the countably infinite set Q of rational numbers in [0, 1]. For each irrational number α in [0, 1], let Q_α be a subset of Q that has α as its unique cluster point. There are uncountably many Q_α's, each of which is countably infinite (and, in particular, not empty). If $Q_\alpha \cap Q_\beta$ is infinite, then it clusters at both α and β, and therefore $\alpha = \beta$.

Second Solution. Let L denote the set of lattice points in the plane: $\{(m,n)|m,n \in Z\}$. Let S_θ be the set of points in L at a distance less than or equal to one from the line $y = \theta x$, $-1 \leqslant \theta \leqslant 1$. Then $|S_{\theta_1} \cap S_{\theta_2}|$ is a finite set if and only if $\theta_1 \neq \theta_2$.

B-5 (17, 17, 1, 1, 0, 0, 0, 1, 17, 29, 37, 79)

Solution. Choose coordinates so that AB and CD are vertical, and so that E lies on the x-axis. For an appropriate slope m we see that B and D lie on the intersection of $y = m(x+d)$ with $x^2 + y^2 = 1$. Thus,

$$y^2 + \left(\frac{y}{m} - d\right)^2 = 1$$

$$\left(1 + \frac{1}{m^2}\right)y^2 - \frac{2d}{m}y + (d^2 - 1) = 0$$

$$y^2 - \left(\frac{2dm}{1+m^2}\right)y + \frac{d^2-1}{1+m^{-2}} = 0.$$

Now (consider y coordinates) $s_1 - s_2$ is *twice* the sum of the roots of this equation, and therefore,

$$s_1 - s_2 = 2\left(\frac{2m}{1+m^2}\right)d \leqslant 2d,$$

with equality if and only if $m = 1$. Thus for $d \neq 0$ the maximum value of $(s_1 - s_2)/d$ is 2 and it is attained if and only if the diagonals intersect at right angles and T is not a square (since $d = 0$ for a square).

B-6 (0, 1, 1, 1, 0, 0, 0, 2, 0, 11, 35, 148)

Solution. The volume in R^n of all points $(x_1, x_2, \dots, x_n)$ with $0 < x_1 < x_2 < \cdots < x_n < 1$ is

$$\int_0^1 \int_0^{x_n} \cdots \int_0^{x_2} dx_1\, dx_2 \dots dx_n = \frac{1}{n!}.$$

Thus, we need to divide

$$M = \sum_{i=0}^{n} \int_0^1 \int_0^{x_n} \cdots \int_0^{x_2} (x_{i+1} - x_i) f(x_{i+1})\, dx_1 \dots dx_n$$

by $1/n!$. When $0 \leqslant i \leqslant n-1$ iterated integration yields for a typical summand

$$\int_0^1 \int_0^{x_n} \cdots \int_0^{x_{i+1}} (x_{i+1} - x_i) \frac{x_i^{i-1}}{(i-1)!} f(x_{i+1})\, dx_i \dots dx_n$$

$$\int_0^1 \int_0^{x_n} \cdots \int_0^{x_{i+2}} \frac{x_{i+1}^{i+1}}{(i+1)!} f(x_{i+1})\, dx_{i+1} \dots dx_n$$

$$\int_0^1 \int_{x_{i+1}}^1 \int_0^{x_n} \cdots \int_{x_{i+1}}^{x_{i+3}} \frac{x_{i+1}^{i+1}}{(i+1)!} f(x_{i+1})\, dx_{i+2} \dots dx_n\, dx_{i+1}$$

$$\int_0^1 \frac{(1 - x_{i+1})^{n-(i+1)}}{(n-(i+1))!} \frac{x_{i+1}^{i+1}}{(i+1)!} f(x_{i+1})\, dx_{i+1}.$$

Multiplication by $n!$ reveals that the kernels are "almost" the terms of the binomial expansion of $((1 - x_{i+1}) + x_{i+1})^n$. In fact by adding in and subtracting out the $i = -1$ term we get

$$n!M = \int_0^1 f(t)\left(1 - (1-t)^n\right) dt + n! J_n,$$

where

$$\begin{aligned} J_n &= \int_0^1 \int_0^{x_n} \cdots \int_0^{x_2} (1 - x_n) f(1)\, dx_1 \dots dx_n \\ &= \int_0^1 \int_0^{x_n} (1 - x_n) \frac{x_{n-1}^{n-2}}{(n-2)!} f(1)\, dx_{n-1}\, dx_n \\ &= \frac{1}{(n+1)!} f(1), \end{aligned}$$

so the expected value is

$$\int_0^1 f(t) P_n(t)\, dt + \frac{1}{n+1} f(1)$$

where $P_n(t) = 1 - (1-t)^n$.