The Fifty-First William Lowell Putnam Mathematical Competition

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The Fifty-First William Lowell Putnam Mathematical Competition

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The following results of the fifty-first William Lowell Putnam Mathematical Competition, held on December 1, 1990, have been determined in accordance with the governing regulations. This annual contest is supported by the William Lowell Putnam Prize Fund for the Promotion of Scholarship, left by Mrs. Putnam in memory of her husband, and is held under the auspices of the Mathematical Association of America.

The first prize, \$5,000, was awarded to the Department of Mathematics of Harvard University. The members of the winning team were: Jordan S. Ellenberg, Raymond M. Sidney, and Eric K. Wepsic; each was awarded a prize of \$250.

The second prize, \$2,500, was awarded to the Department of Mathematics of Duke University. The members of the winning team were Jeanne A. Nielsen, Will A. Schneeberger, and Jeffrey M. Vanderkam; each was awarded a prize of \$200.

The third prize, \$1,500, was awarded to the Department of Mathematics of the University of Waterloo. The members of the winning team were Dorian Birsan, Daniel R. L. Brown, and Colin M. Springer; each was awarded a prize of \$150.

The fourth prize, \$1,000, was awarded to the Department of Mathematics of Yale University. The members of the winning team were Thomas Zuwei Feng, Andrew H. Kresch, and Zhaoliang Zhu; each was awarded a prize of \$100.

The fifth prize, \$500, was awarded to the Department of Mathematics of Washington University. The members of the winning team were William Chen, Adam M. Costello, and Jordan A. Samuels; each was awarded a prize of \$50.

The five highest ranking individual contestants, in alphabetical order, were Jordan S. Ellenberg, Harvard University; Jordan Lampe, University of California, Berkeley; Raymond M. Sidney, Harvard University; Ravi D. Vakil, University of Toronto; and Eric K. Wepsic, Harvard University. Each of these was designated a Putnam Fellow by the Mathematical Association of America and awarded a prize of \$500 by the Putnam Prize Fund.

The next seven highest ranking individuals, in alphabetical order, were Andrew H. Kresch, Yale University; Samuel A. Kutin, Harvard University; Royce Y. Peng, Harvard University; Eric M. Rains, Case Western Reserve University; Jeffrey M. Vanderkam, Duke University; Samuel K. Vandervelde, Swarthmore College; and Michael E. Zieve, Harvard University. Each was awarded a prize of \$250.

The following teams, named in alphabetical order, received honorable mention: California Institute of Technology, with team members Tien-Yee Chiu, Russell A. Manning, and Robert G. Southworth; the University of California, Berkeley, with team members Brian J. Birgen, Jordan Lampe, and Thomas S. Lumley; Massachusetts Institute of Technology, with team members Christos Athanasiadis, Andrew Chou, and David B. Wilson; Stanford University, with team members, Daniel P. Cory, Gregory G. Martin, and Andras Vasy; and Swarthmore College, with team members Olaf A. Holt, Robert E. Marx, and Samuel K. Vandervelde.

Honorable mention was achieved by the following thirty-five individuals named in alphabetical order: Eric M. Boesch, University of Maryland, College Park;

Hubert L. Bray, Rice University; Daniel R. L. Brown, University of Waterloo; Michael J. Callahan, Harvard University; David B. Carlton, Harvard University; Helmut R. Celina, Davidson College; Nickolai I. Chavdarov, Brandeis University; William Chen, Washington University, St. Louis; Mark T. Chrisman, University of California, Davis; Bryan F. Clair, University of California, Berkeley; Brian D. Conrad, Harvard University; Daniel B. Finn, University of Rochester; Joshua B. Fischman, Princeton University; Mikhail Grinberg, Massachusetts Institute of Technology; Alex Gurevich, University of Maryland, College Park; Richard S. Kiss, Simon Fraser University; John C. Loftin, Stanford University; Gregory G. Martin, Stanford University; David K. McKinnon, Harvard University; Jeanne A. Nielson, Duke University; David M. Patrick, Carnegie Mellon University; Alexander R. Pruss, University of Western Ontario; Jordan A. Samuels, Washington University, St. Louis; Will A. Schneeberger, Duke University; Lawren M. Smithline, Harvard University; Robert G. Southworth, California Institute of Technology; Colin M. Springer, University of Waterloo; Jun Teng, California Institute of Technology; Andras Vasy, Stanford University; Martin M. Wattenberg, Brown University; David B. Wilson, Massachusetts Institute of Technology; Michael P. Wolf, Harvard University; and John H. Woo, Harvard University.

The other individuals who achieved ranks among the top 98, in alphabetical order of their schools, were: Boston University, Michael G. Szydlo; University of British Columbia, Gregory F. Wellman; University of Calgary, Geoffrey T. Falk; California Institute of Technology, Ian Agol, Alan I. Knutson, William M. Watson; University of California, Berkeley, Stephen P. Bard, Thomas S. Lumley, Max L. Shireson, Zheng Yin; Carleton University, Adam M. Logan; Carnegie Mellon University, Sanjay Khanna; University of Chicago, David J. Pollack, Adrian Tanner; Columbia University, Andrew Mogilyansky; Cornell University, Isaac J. Kuo; Harvard University, Daniel E. Gottesman, F. Dean Hildebrandt, Roger W. Lee, Andrew P. Lewis; Harvey Mudd College, Guy D. Moore; University of Illinois, Urbana-Champaign, David E. Bekman; University of Maryland, College Park, Lev Novik; Massachusetts Institute of Technology, Andrew Chou, Edward B. Hontz, Michael J. Lawler; Michigan State University, Thomas P. Hayes, Jacob R. Lorch; University of Minnesota, Twin Cities, Wei Shen; University of Missouri, Rolla, Xi Chen; Mount Allison University, Eugene Fink; New York University, Daniel J. Bernstein; State University of New York, Stony Brook, Jason Israel; Princeton University, Timothy Y. Chow, Gregory D. Landweber; Queens University, Alex Grossman; Reed College, Nathaniel J. Thurston; Stanford University, Beesham A. Seecharan, Jay A. Shrauner, Garrett R. Vargas; Swarthmore College, Olaf A. Holt; University of Texas, Austin, Bryan W. Taylor; University of Toronto, Nima Arkani-Hamed; Trinity College, Hartford, Marshall A. Whittlesey; Washington State University, Julie B. Kerr; Washington University, St. Louis, Peter H. Berman, Adam M. Costello, Jeremy T. Tyson; University of Waterloo, Michael A. Buckley, John Daniel Christensen; and Yale University, Thomas Zuwei Feng, Evan M. Gilbert, Zhaoliang Zhu.

There were 2347 individual contestants from 380 colleges and universities in Canada and the United States in the competition of December 1, 1990. Teams were entered by 289 institutions.

The Questions Committee for the fifty-first competition consisted of George E. Andrews, Paul R. Halmos (Chair), and Kenneth A. Stolarsky; they composed the problems listed below and were most prominent among those suggesting solutions.

PROBLEMS

Problem A-1. Let

$$T_0 = 2, \qquad T_1 = 3, \qquad T_2 = 6,$$

and for $n \geq 3$,

$$T_n = (n+4)T_{n-1} - 4nT_{n-2} + (4n-8)T_{n-3}.$$

The first few terms are

$$2, 3, 6, 14, 40, 152, 784, 5168, 40576, 363392.$$

Find, with proof, a formula for T_n of the form $T_n = A_n + B_n$, where (A_n) and (B_n) are well-known sequences.

Problem A-2. Is $\sqrt{2}$ the limit of a sequence of numbers of the form $\sqrt[3]{n} - \sqrt[3]{m}$, $(n, m = 0, 1, 2, \ldots)$?

Problem A-3. Prove that any convex pentagon whose vertices (no three of which are collinear) have integer coordinates must have area $\geq 5/2$.

Problem A-4. Consider a paper punch that can be centered at any point of the plane and that, when operated, removes from the plane precisely those points whose distance from the center is irrational. How many punches are needed to remove every point?

Problem A-5. If $\mathbf{A}$ and $\mathbf{B}$ are square matrices of the same size such that $\mathbf{ABAB} = \mathbf{0}$, does it follow that $\mathbf{BABA} = \mathbf{0}$?

Problem A-6. If X is a finite set, let $|X|$ denote the number of elements in X. Call an ordered pair (S, T) of subsets of $\{1, 2, \ldots, n\}$ *admissible* if $s > |T|$ for each $s \in S$, and $t > |S|$ for each $t \in T$. How many admissible ordered pairs of subsets of $\{1, 2, \ldots, 10\}$ are there? Prove your answer.

Problem B-1. Find all real-valued continuously differentiable functions f on the real line such that for all x

$$(f(x))^2 = \int_0^x \left((f(t))^2 + (f'(t))^2\right) dt + 1990.$$

Problem B-2. Prove that for $|x| < 1$, $|z| > 1$,

$$1 + \sum_{j=1}^{\infty} (1 + x^j) \frac{(1-z)(1-zx)(1-zx^2)\cdots(1-zx^{j-1})}{(z-x)(z-x^2)(z-x^3)\cdots(z-x^j)} = 0.$$

Problem B-3. Let S be a set of 2×2 integer matrices whose entries a_{ij} (1) are all squares of integers, and, (2) satisfy $a_{ij} \leq 200$. Show that if S has more than 50387 $(= 15^4 - 15^2 - 15 + 2)$ elements, then it has two elements that commute.

Problem B-4. Let G be a finite group of order n generated by a and b. Prove or disprove: there is a sequence

$$g_1, g_2, g_3, \ldots g_{2n}$$

such that

(1) every element of G occurs exactly twice, and

(2) g_{i+1} equals $g_i a$ or $g_i b$, for $i = 1, 2, \ldots, 2n$. (Interpret g_{2n+1} as g_1.)

Problem B-5. Is there an infinite sequence $a_0, a_1, a_2, \ldots$ of nonzero real numbers such that for $n = 1, 2, 3, \ldots$ the polynomial

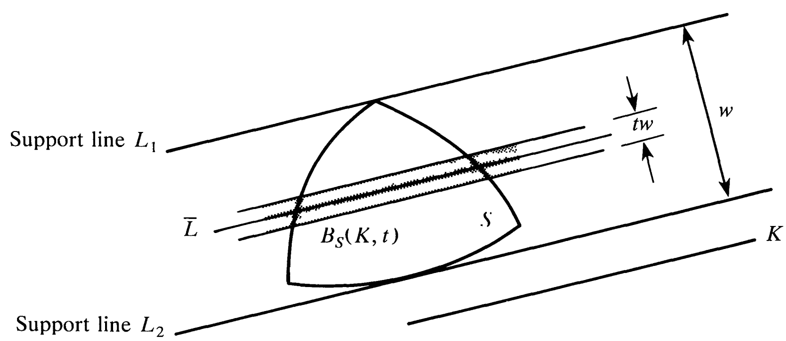
$$p_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

has exactly n distinct real roots?

Problem B-6. Let S be a nonempty closed bounded convex set in the plane. Let K be a line and t a positive number. Let L_1 and L_2 be support lines for S parallel to K, and let $\overline{L}$ be the line parallel to K and midway between L_1 and L_2. Let $B_S(K, t)$ be the band of points whose distance from $\overline{L}$ is at most $(t/2)w$, where w is the distance between L_1 and L_2. What is the smallest t such that

$$S \cap \bigcap_K B_S(K, t) \neq \emptyset$$

for all S? (K runs over all lines in the plane.)



SOLUTIONS

In the 12-tuples $(n_{10}, n_9, \ldots, n_0, n_{-1})$ following each problem number below, n_i for $10 \geq i \geq 0$ is the number of students among the top 199 contestants achieving i points for the problem and n_{-1} is the number of those not submitting solutions.

A-1 (150, 9, 1, 0, 0, 0, 0, 0, 1, 1, 6, 33)

Solution. The formula for T_n is

$$T_n = n! + 2^n.$$

This can be verified by induction. Alternatively, set $t_n = n! + 2^n$. Clearly $t_0 = 2 = T_0$, $t_1 = 3 = T_1$ and $t_2 = 6 = T_2$. Also,

$$t_n - n t_{n-1} = 2^n - n 2^{n-1}.$$

Now 2^n and $n2^{n-1}$ are both solutions of the recurrence equation

$$f_n - 4f_{n-1} + 4f_{n-2} = 0, \qquad (*)$$

which is easily shown by direct substitution. Therefore since $t_n - nt_{n-1}$ is a linear combination of solutions to $(*)$, it must also be a solution. Consequently,

$$(t_n - nt_{n-1}) - 4(t_{n-1} - (n-1)t_{n-2}) + 4(t_{n-2} - (n-2)t_{n-3}) = 0,$$

or

$$t_n = (n+4)t_{n-1} - 4nt_{n-2} + (4n-8)t_{n-3}.$$

Hence $t_n = T_n$ because they are identical for $n = 0, 1, 2$ and satisfy the same third-order recurrence $(*)$ for $n \geq 3$.

A-2 (63, 25, 16, 4, 0, 0, 0, 4, 5, 6, 21, 57)

Solution. Since

$$\sqrt[3]{n+1} - \sqrt[3]{n} = \frac{1}{\sqrt[3]{(n+1)^2} + \sqrt[3]{(n+1)n} + \sqrt[3]{n^2}},$$

it follows that $\sqrt[3]{n+1} - \sqrt[3]{n} \to 0$ as $n \to \infty$, and hence that there are arbitrarily small numbers of the form $\sqrt[3]{n} - \sqrt[3]{m}$. Since $k(\sqrt[3]{n} - \sqrt[3]{m}) = \sqrt[3]{k^3 n} - \sqrt[3]{k^3 m}$, it follows that the set of numbers of that form is closed under multiplication by arbitrary positive integers. The preceding two sentences imply that the set of numbers of the form under consideration is dense, and hence that every real number is a limit of a sequence of such numbers.

A-3 (4, 4, 4, 0, 0, 0, 0, 0, 22, 36, 76, 55)

Solution. By Pick's formula, the area is $I + B/2 - 1$, where I is the number of internal lattice points and B is the number on the boundary. Clearly, $I \geq 0$ and $B \geq 5$. If $I \geq 1$ we are done.

If $I = 0$, then separate the vertices v_1, v_2, v_3, v_4, v_5 into four classes according to the parity of their coordinates. At least one class must have at least two elements, say v_1 and v_2. Hence the mid-point $\frac{1}{2}(v_1 + v_2)$, is also a lattice point; call it v_0. Since $I = 0$, v_0 is on the boundary of the pentagon. Now consider the five points $\{v_0, v_2, v_3, v_4, v_5\}$. The same reasoning produces a second lattice point v_0^* which is not v_1 (since v_0^* is a mid-point) and not in the interior (since $I = 0$.) Thus we have a second new lattice point on the boundary. Therefore, $B \geq 7$, so again the area is $\geq 5/2$.

A-4 (44, 7, 6, 6, 0, 0, 0, 0, 32, 15, 30, 61)

Solution. The answer is certainly greater than 2. Reason: to any two distinct points there corresponds at least one point whose distance from each of the given ones is rational. Proof: draw circles centered at the given points with rational radii; if the circles are not chosen too carelessly, they will intersect. (Choose the radii to be more than half the given distance but less than the whole.)

Three punches are enough. Indeed: punch twice, at distinct centers. Since each punch leaves countably many circles, the two punches leave their intersections, a countable set. Consider all circles centered at points of that set, with rational radii; their intersections with an arbitrary line form a countable set. A point of that line not in that countable set is at an irrational distance from all remaining points; apply the punch there.

A-5 (29, 5, 0, 0, 0, 0, 0, 0, 1, 0, 58, 108)

Solution. The answer is yes for 2×2 matrices and no in all other cases. Indeed: since $\mathbf{ABAB} = \mathbf{0}$, it follows that $\mathbf{B(ABAB)A} = \mathbf{0}$, and hence that $\mathbf{BA}$ is nilpotent. If a 2×2 matrix $\mathbf{M}$ is nilpotent, then $\mathbf{M}^2 = 0$ (because the characteristic equation of $\mathbf{M}$ has degree 2 or less).

A counterexample for 3×3 (and therefore, just by enlargement by 0's, for any size) is to take

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

A-6 (6, 6, 54, 1, 0, 0, 0, 0, 4, 0, 45, 85)

Solution. Let A_n denote the number of admissible ordered pairs of subsets of $\{1, 2, \dots, n\}$. Clearly

$$A_n = \sum_{0 \le i, j \le n} \binom{n-i}{j}\binom{n-j}{i}.$$

Define

$$B_n = \sum_{0 \le i, j \le n} \binom{n+1-i}{j}\binom{n-j}{i}.$$

Clearly $A_0 = B_0 = 1$, and

$$\begin{aligned} B_n - A_n &= \sum_{0 \le i, j \le n} \binom{n-i}{j-1}\binom{n-j}{i} \\ &= \sum_{0 \le i, j \le n-1} \binom{n-i}{j}\binom{n-1-j}{i} \\ &= B_{n-1}, \end{aligned}$$

while

$$\begin{aligned} A_n - B_{n-1} &= \sum_{0 \le i, j \le n-1} \binom{n-i}{j}\binom{n-j-1}{i-1} + 2 \\ &= \sum_{0 \le i, j \le n-1} \binom{n-i-1}{j}\binom{n-j-1}{i} + 1 \\ &= A_{n-1} + 1. \end{aligned}$$

Hence we immediately verify by induction that

$$A_n = F_{2n+2}, \qquad B_n = F_{2n+3} - 1.$$

Hence, $A_{10} = F_{22} = 17711$. (F_i is the i-th Fibonacci number, defined by $F_0 = 0$, $F_1 = 1$, and for $n \ge 2$, $F_n = F_{n-1} + F_{n-2}$.)

B-1 (114, 2, 52, 0, 0, 0, 0, 11, 5, 3, 10, 4)

Solution. There are two such functions, namely $f(x) = \sqrt{1990}\, e^x$, and $f(x) = -\sqrt{1990}\, e^x$. To see this, suppose that the identity holds. Differentiating each side gives

$$2f(x)f'(x) = (f(x))^2 + (f'(x))^2,$$

or equivalently,

$$(f(x) - f'(x))^2 = 0, \qquad f'(x) = f(x),$$
$$\log|f(x)| = x + C, \qquad |f(x)| = e^C e^x.$$

But f is continuous and $f(0) = \pm\sqrt{1990}$, and this implies that $f(x) = \pm\sqrt{1990}\, e^x$.

B-2 (23, 5, 4, 9, 0, 0, 0, 3, 0, 0, 32, 125)

Solution. Let $S_0 = 1$, and for $n \geq 1$, let

$$S_n = 1 + \sum_{j=1}^{n} (1 + x^j)\frac{(1-z)(1-zx)(1-zx^2)\cdots(1-zx^{j-1})}{(z-x)(z-x^2)(z-x^3)\cdots(z-x^j)}.$$

It is easy to check that $S_1 = (1-zx)/(z-x)$, $S_2 = (1-zx)(1-zx^2)/(z-x)(z-x^2)$, and by induction,

$$S_n = \frac{(1-zx)(1-zx^2)\cdots(1-zx^n)}{(z-x)(z-x^2)\cdots(z-x^n)}.$$

To complete the proof, we need to prove that $\lim_{n\to\infty} S_n = 0$. To see this, we note that

$$S_{n+1} = \left(\frac{1-zx^{n+1}}{z-x^{n+1}}\right)S_n.$$

As $n \to \infty$, $1 - zx^{n+1}$ goes to 1 and $z - x^{n+1}$ goes to z. Thus, there exist positive numbers N and ε such that

$$\left|\frac{1-zx^{n+1}}{z-x^{n+1}}\right| < \frac{1}{|z|} + \varepsilon < 1$$

for all integers $n > N$. It follows that

$$|S_{n+1}| < \left(\frac{1}{|z|} + \varepsilon\right)|S_n|$$

and the result follows.

B-3 (97, 7, 4, 2, 0, 0, 0, 0, 12, 2, 54, 23)

Solution. Let U be all such 2×2 matrices, D the diagonal ones, and J those that are multiples of $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Note that (i) any two from D commute, (ii) any two from J commute, and (iii) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$ commute. Clearly,

$$|U \cap (D \cup J)^c| = |U| - |D| - |J| + |D \cap J| = 15^4 - 15^2 - 15 + 1.$$

Suppose that no two elements of S commute, and write

$$S = (S \cap (D \cup J)) \cup (S \cap (D \cup J)^c).$$

Clearly, $|S \cap (D \cup J)| \leq 2$ and (by (iii) above)

$$|S \cap (D \cup J)^c| < |U \cap (D \cup J)^c| \leq 15^4 - 15^2 - 15 + 1.$$

The result follows. (Here X^c denotes the complement of X.)

The number 50387 is far from the best possible and there are many potential solutions. *Jiuqiang Liu* and *Allen J. Schwenk* from *Western Michigan University* have shown, using an inclusion-exclusion argument, that the maximum number of elements in U in which no two elements commute is 32390.

B-4 (9, 5, 2, 1, 0, 0, 0, 0, 3, 2, 63, 116)

Solution. Construct a graph whose vertices are labeled by the elements of G, so that for each vertex g, there are two "out" arcs, one to vertex ga and one to vertex gb (and consequently, each vertex g has two "in" arcs coming to it, one from ga^{-1} and one from gb^{-1}). The resulting graph is connected and each vertex has outdegree 2 and indegree 2. Therefore there is an Eulerian path which traverses the arcs, once and only once, and returns to the beginning. We get the desired sequence by listing the group elements associated with the vertices as we follow this path.

B-5 (16, 11, 15, 12, 0, 0, 0, 11, 5, 5, 48, 78)

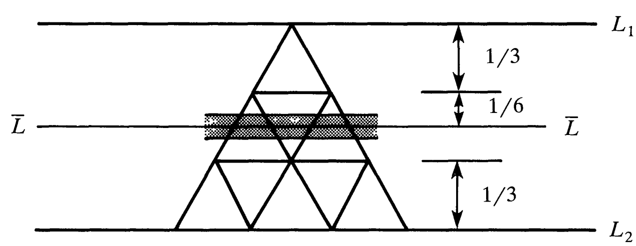
Solution. Take $a_0 = 1$, $a_1 = -1$, and proceed by induction. Say $p_n(x)$ has the property, and also $p_n(x) \to \infty$ or $-\infty$ as $x \to \infty$ depending upon whether n is even or odd. Then

$$p_{n+1}(x) \equiv p_n(x) + \frac{(-x)^{n+1}}{M}$$

has a sign change arbitrarily close to every root of $p_n(x)$ for M sufficiently large, and also the same sign as $p_n(x)$ at $x_n^* + 1$ where x_n^* is the largest root of $p_n(x)$. But now (M is already fixed) for x sufficiently large $p_{n+1}(x)$ has another sign change. Since $p_{n+1}(x)$ has at most $n + 1$ roots, the result follows.

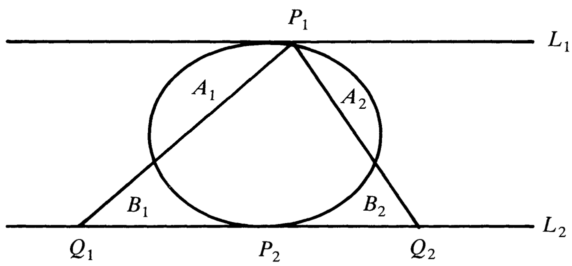
B-6 (5, 0, 0, 0, 0, 0, 0, 0, 38, 6, 29, 123)

Solution. Consider the dissection of the equilateral triangle (as shown) into 9 similar equilateral triangles.



It shows that any $t < 1/3$ produces an empty intersection.

We now show that the intersection is nonempty for $t \geq 1/3$, since it always contains the centroid of S. Think of L_1 as the upper support line (see sketch).



Let P_1 be a point of contact of L_1 with S, and P_2 a point of contact of L_2 with S. Extend two lines from P_1 to points Q_1, Q_2 on L_2 so that A_1, the region "above" P_1Q_1 and in S has area equal to that of B_1, the region "below" P_1Q_1 above L_2 and outside of S. If S is perturbed by replacing A_1 by B_1 (and similarly A_2 by B_2) the new S (it is a triangle) will have a "lower" centroid. But this new centroid is still $1/3$ of the way above L_2 (on the way to L_1). Hence if β is a band comprising the middle $1/3$ of the strip between L_1 and L_2, it contains the centroid of S. Hence for $t = 1/3$, the intersection is nonempty. Hence $t = 1/3$ is the smallest such value.