The Fifty-Second William Lowell Putnam Mathematical Competition

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The following results of the fifty-second William Lowell Putnam Mathematical Competition, held on December 7, 1991, have been determined in accordance with the governing regulations. This annual contest is supported by the William Lowell Putnam Prize Fund for the Promotion of Scholarship, left by Mrs. Putnam in memory of her husband, and is held under the auspices of the Mathematical Association of America.

The first prize, \$5,000, was awarded to the Department of Mathematics of Harvard University. The members of the winning team were: Jordan S. Ellenberg, Samuel A. Kutin, and Eric K. Wepsic; each was awarded a prize of \$250.

The second prize, \$2,500, was awarded to the Department of Mathematics of the University of Waterloo. The members of the winning team were Daniel R. L. Brown, Ian A. Goldberg, and Colin M. Springer; each was awarded a prize of \$200.

The third prize, \$1,500, was awarded to the Department of Mathematics of Harvey Mudd College. The members of the winning team were Timothy P. Kokesh, Jon H. Leonard, and Guy D. Moore; each was awarded a prize of \$150.

The fourth prize, \$1,000, was awarded to the Department of Mathematics of Stanford University. The members of the winning team were Gregory G. Martin, Garrett R. Vargas, and András Vasy; each was awarded a prize of \$100.

The fifth prize, \$500, was awarded to the Department of Mathematics of Yale University. The members of the winning team were Zuwei Thomas Feng, Evan M. Gilbert, and Andrew H. Kresch; each was awarded a prize of \$50.

The five highest ranking individual contestants, in alphabetical order, were Xi Chen, University of Missouri, Rolla; Joshua B. Fischman, Princeton University; Samuel A. Kutin, Harvard University; Ravi D. Vakil, University of Toronto; and Eric K. Wepsic, Harvard University. Each of these was designated a Putnam Fellow by the Mathematical Association of America and awarded a prize of \$500 by the Putnam Prize Fund.

The next five highest ranking individuals, in alphabetical order, were Daniel R. L. Brown, University of Waterloo; Gregory G. Martin, Stanford University; David M. Patrick, Carnegie Mellon University; Jun Teng, California Institute of

Technology; and Jeffrey M. Vanderkam, Duke University. Each was awarded a prize of $250.

The following teams, named in alphabetical order, received honorable mention: the University of British Columbia, with team members Rob M. Deary, Malik M. Kalfane, and Mark A. Van Raamsdonk; the Massachusetts Institute of Technology, with team members Christos Athanasiadis, Henry L. Cohn, and Mikhail Grinberg; Oberlin College, with team members Gary N. Felder, Susan J. Patterson, and Ian B. Robertson; Princeton University, with team members Joshua B. Fischman, Peter R. Kramer, and Gregory D. Landweber; and the University of Toronto, with team members Nima Arkani-Hamed, Jeff T. Higham, and Ravi D. Vakil.

Honorable mention was achieved by the following thirty-four individuals named in alphabetical order: Christos Athanasiadis, Massachusetts Institute of Technology; Radu Bacioiu, Dartmouth College; David S. Bigham, Duke University; Hubert L. Bray, Rice University; Daniel P. Cory, Stanford University; Graham C. Denham, University of Alberta; Jordan S. Ellenberg, Harvard University; Ian A. Goldberg, University of Waterloo; Steven S. Gubser, Princeton University; F. Dean Hildebrandt, Harvard University; Daniel C. Isaksen, University of California, Berkeley; Dmitry A. Ivanov, Georgia Institute of Technology; Timothy P. Kokesh, Harvey Mudd College; Andrew H. Kresch, Yale University; Gregory D. Landweber, Princeton University; Roger W. Lee, Harvard University; Andrew P. Lewis, Harvard University; Jacob R. Lorch, Michigan State University; Samuel J. Maltby, University of Calgary; David K. McKinnon, Harvard University; Peter L. Milley, University of Waterloo; Guy D. Moore, Harvey Mudd College; Demetrio A. Muñoz, Cornell University; Lev Novik, University of Maryland, College Park; Joel E. Rosenberg, Princeton University; Colin M. Springer, University of Waterloo; Andrej Šuch, Queen's University; Dylan P. Thurston, Harvard University; Samuel K. Vandervelde, Swarthmore College; Garrett R. Vargas, Stanford University; Kevin M. Wald, Harvard University; Erick Wong, Simon Fraser University; John H. Woo, Harvard University; and Michael E. Zieve, Harvard University.

The other individuals who achieved ranks among the top 100, in alphabetical order of their schools, were: Boston University, Michael G. Szydlo; University of British Columbia, Rob M. Deary, Mark A. Van Raamsdonk; Brown University, Kenneth W. Bromberg; California Institute of Technology, William M. Watson; University of California, Berkeley, Benjamin J. Davis; University of California, Los Angeles, Christopher B. Baker; Carleton College, Mark J. Logan; Dartmouth College, Paul B. Larson, Dan O. Popa; Duke University, David M. Jones; Harvard University, David B. Carlton, Tal N. Kubo, Lawren M. Smithline; Harvey Mudd College, Jon H. Leonard; Hope College, Alexey G. Stepanov; University of Illinois, Champaign-Urbana, David E. Beckman; Kalamazoo College, Kenneth P. Mulder; Le Tourneau University, Bryan D. Greer; Massachusetts Institute of Technology, Thomas C. Chou, Henry L. Cohn, Michael J. Lawlor, Patrick J. LoPresti, Todd W. Rowland, Jason M. Sachs, David E. Tang; Michigan State University, Thomas P. Hayes; University of Michigan, Ann Arbor, Soundararajan Kannan; New York University, David P. Gamarnik; Northwestern University, Ashvin M. Sangoram; Oberlin College, Ian B. Robertson; University of Pennsylvania, Frosti Petursson; Princeton University, Ze-Yu Chen, Jonathan T. Higa, Adam M. Logan, Mark W. Lucianovic; Rice University, Clark B. Bray; University of Rochester, Daniel B. Finn; Rose Hulman Institute of Technology, Jonathan E. Atkins; Stanford University, James M. Mailhot; Swarthmore College, David A. Packer; University of Texas, Austin, Douglas S. Hauge; University of Toronto, Jeff T. Higham, Colin J.

Rust, Hugh A. Thomas; Trinity College, Hartford, Marshall A. Whittlesey; University of Victoria, Benjamin J. Tilly; Washington State University, Julie B. Kerr; Washington University, St. Louis, Scott P. Nudelman, Jeremy T. Strzynski; University of Waterloo, Paul L. Check, James H. Coleman, Jie J. Lou; Yale University, Zuwei Thomas Feng, Matthew Frank, Zhaohui Zhang.

There were 2325 individual contestants from 383 colleges and universities in Canada and the United States in the competition of December 7, 1991. Teams were entered by 291 institutions.

The Questions Committee for the fifty-second competition consisted of George E. Andrews, George T. Gilbert, and Kenneth A. Stolarsky (Chair); they composed the problems listed below and were most prominent among those suggesting solutions.

PROBLEMS

Problem A-1.

A 2×3 rectangle has vertices at $(0,0)$, $(2,0)$, $(0,3)$, and $(2,3)$. It rotates $90°$ clockwise about the point $(2,0)$. It then rotates $90°$ clockwise about the point $(5,0)$, then $90°$ clockwise about the point $(7,0)$, and finally, $90°$ clockwise about the point $(10,0)$. (The side originally on the x-axis is now back on the x-axis.) Find the area of the region above the x-axis and below the curve traced out by the point whose initial position is $(1,1)$.

Problem A-2.

Let $\mathbf{A}$ and $\mathbf{B}$ be different $n \times n$ matrices with real entries. If $\mathbf{A}^3 = \mathbf{B}^3$ and $\mathbf{A}^2\mathbf{B} = \mathbf{B}^2\mathbf{A}$, can $\mathbf{A}^2 + \mathbf{B}^2$ be invertible?

Problem A-3.

Find all real polynomials $p(x)$ of degree $n \geq 2$ for which there exist real numbers $r_1 < r_2 < \cdots < r_n$ such that

$$\text{(i)} \quad p(r_i) = 0, \qquad i = 1, 2, \ldots, n,$$

and

$$\text{(ii)} \quad p'\left(\frac{r_i + r_{i+1}}{2}\right) = 0, \qquad i = 1, 2, \ldots, n-1,$$

where $p'(x)$ denotes the derivative of $p(x)$.

Problem A-4.

Does there exist an infinite sequence of closed discs $D_1, D_2, D_3, \ldots$ in the plane, with centers $c_1, c_2, c_3, \ldots$, respectively, such that

(i) the c_i have no limit point in the finite plane,
(ii) the sum of the areas of the D_i is finite, and
(iii) every line in the plane intersects at least one of the D_i?

Problem A-5.

Find the maximum value of

$$\int_0^y \sqrt{x^4 + (y - y^2)^2}\, dx$$

for $0 \le y \le 1$.

Problem A-6.

Let $A(n)$ denote the number of sums of positive integers $a_1 + a_2 + \cdots + a_r$ that add up to n with $a_1 > a_2 + a_3$, $a_2 > a_3 + a_4, \ldots, a_{r-2} > a_{r-1} + a_r$, $a_{r-1} > a_r$. Let $B(n)$ denote the number of $b_1 + b_2 + \cdots + b_s$ that add up to n, with

(i) $b_1 \ge b_2 \ge \cdots \ge b_s$,
(ii) each b_i is in the sequence $1, 2, 4, \ldots, g_j, \ldots$ defined by $g_1 = 1$, $g_2 = 2$, and $g_j = g_{j-1} + g_{j-2} + 1$, and
(iii) if $b_1 = g_k$ then every element in $\{1, 2, 4, \ldots, g_k\}$ appears at least once as a b_i.

Prove that $A(n) = B(n)$ for each $n \ge 1$.

(For example, $A(7) = 5$ because the relevant sums are 7, 6 + 1, 5 + 2, 4 + 3, 4 + 2 + 1, and $B(7) = 5$ because the relevant sums are 4 + 2 + 1, 2 + 2 + 2 + 1, 2 + 2 + 1 + 1 + 1, 2 + 1 + 1 + 1 + 1 + 1, 1 + 1 + 1 + 1 + 1 + 1 + 1.)

Problem B-1.

For each integer $n \ge 0$, let $S(n) = n - m^2$, where m is the greatest integer with $m^2 \le n$. Define a sequence $(a_k)_{k=0}^{\infty}$ by $a_0 = A$ and $a_{k+1} = a_k + S(a_k)$ for $k \ge 0$. For what positive integers A is this sequence eventually constant?

Problem B-2.

Suppose f and g are nonconstant, differentiable, real-valued functions on $\mathbf{R}$. Furthermore, suppose that for each pair of real numbers x and y,

$$f(x+y) = f(x)f(y) - g(x)g(y),$$

$$g(x+y) = f(x)g(y) + g(x)f(y).$$

If $f'(0) = 0$, prove that $(f(x))^2 + (g(x))^2 = 1$ for all x.

Problem B-3.

Does there exist a real number L such that, if m and n are integers greater than L, then an $m \times n$ rectangle may be expressed as a union of 4×6 and 5×7 rectangles, any two of which intersect at most along their boundaries?

Problem B-4.

Suppose p is an odd prime. Prove that

$$\sum_{j=0}^{p} \binom{p}{j}\binom{p+j}{j} \equiv 2^p + 1 \pmod{p^2}.$$

Problem B-5.

Let p be an odd prime and let $\mathbf{Z}_p$ denote (the field of) the integers modulo p. How many elements are in the set

$$\{x^2 : x \in \mathbf{Z}_p\} \cap \{y^2 + 1 : y \in \mathbf{Z}_p\}?$$

Problem B-6.

Let a and b be positive numbers. Find the largest number c, in terms of a and b, such that

$$a^x b^{1-x} \le a\frac{\sinh ux}{\sinh u} + b\frac{\sinh u(1-x)}{\sinh u}$$

for all u with $0 < |u| \le c$ and for all x, $0 < x < 1$. (Note: $\sinh u = (e^u - e^{-u})/2$.)

SOLUTIONS

In the 12-tuples $(n_{10}, n_9, \ldots, n_0, n_{-1})$ following each problem number below, n_i for $10 \ge i \ge 0$ is the number of students among the top 213 contestants achieving i points for the problem and n_{-1} is the number of those not submitting solution.

A-1 (189, 0, 3, 0, 0, 0, 0, 0, 0, 1, 20, 0)

Solution. The point $(1,1)$ rotates around $(2,0)$ to $(3,1)$, then around $(5,0)$ to $(6,2)$, then around $(7,0)$ to $(9,1)$, then around $(10,0)$ to $(11,1)$. The area of concern consists of four 1×1 right triangles of area $1/2$, four 1×2 right triangles of area 1, two quarter circles of area $(\pi/4)(\sqrt{2})^2 = \pi/2$, and two quarter circles of area $(\pi/4)(\sqrt{5})^2 = 5\pi/4$. Hence the total area is $7\pi/2 + 6$.

A-2 (150, 17, 2, 1, 0, 0, 0, 0, 3, 5, 15, 20)

Solution. No. If so, then $\mathbf{A} - \mathbf{B} = (\mathbf{A}^2 + \mathbf{B}^2)^{-1}(\mathbf{A}^2 + \mathbf{B}^2)(\mathbf{A} - \mathbf{B}) = (\mathbf{A}^2 + \mathbf{B}^2)^{-1}(\mathbf{A}^3 + \mathbf{B}^2\mathbf{A} - \mathbf{A}^2\mathbf{B} - \mathbf{B}^3) = (\mathbf{A}^2 + \mathbf{B}^2)^{-1}\mathbf{0} = \mathbf{0}$, so $\mathbf{A} = \mathbf{B}$, a contradiction.

A-3 (42, 35, 29, 0, 0, 0, 0, 0, 6, 5, 63, 33)

Solution. The set of polynomials is $\{ax^2 + bx + c : a \ne 0,\ b^2 - 4ac > 0\}$.

First, if $p(x)$ is such a polynomial, it must have two distinct real roots, say r_1, r_2, with $r_1 < r_2$. It is easy to check that such polynomials meet the condition. To show nothing else does, write

$$p(x) = a(x - r_1)(x - r_2) \cdots (x - r_n)$$

where $r_1 < r_2 < \cdots < r_n$ and $n \geq 3$. Then

$$p'(x) = a(2x - (r_1 + r_2))q(x) + a(x - r_1)(x - r_2)q'(x),$$

where $q(x) = (x - r_3)\cdots(x - r_n)$. By Rolle's Theorem, all the zeros of $q'(x)$ lie between r_3 and r_n. Hence $(r_1 + r_2)/2$ is not a zero of $q'(x)$, showing that $p(x)$ does not meet the condition.

A-4 (86, 33, 43, 0, 0, 0, 0, 0, 12, 3, 21, 15)

Solution. Let a_i be a decreasing sequence of positive numbers $a_1 \leq 1$, $\Sigma a_i = \infty$, and $\Sigma a_i^2 < \infty$ (for example, $a_i = 1/i$). Let D_i be a disc of radius a_i. Cover $x^2 + y^2 = 1$ by translates (each of which shall intersect $x^2 + y^2 = 1$) of $D_1, D_2, \ldots, D_{m_1}$ with $m_1 < \infty$. This can be done since $\Sigma\,\mathrm{diam}(D_i) = 2\Sigma a_i = \infty$.

Now cover $x^2 + y^2 = 2$ similarly by translates of $D_{m_1+1}, \ldots, D_{m_2}$ where $m_2 < \infty$ (same justification), $\ldots$, $x^2 + y^2 = k$ by $D_{m_{k-1}+1}, \ldots, D_{m_k}$, etc.

Clearly, every line intersects $x^2 + y^2 = k$ for some integer k; moreover, $\Sigma a_i^2 < \infty$ implies $\Sigma\,\mathrm{area}(D_i) = \pi\Sigma a_i^2$ is finite.

Finally, any disc is inside of a disc $x^2 + y^2 = k_0$, and the discs covering $x^2 + y^2 \leq h$ for $h > k_0 + 4$ cannot intersect $x^2 + y^2 \leq k_0$ (recall (a_i) is decreasing, $a_i \leq 1$). Hence the c_i have no limit point, since no disc may contain infinitely many of them.

A-5 (23, 4, 5, 0, 0, 0, 0, 0, 3, 6, 82, 90)

Solution. For $0 \leq y \leq 1$ let $I(y) = \int_0^y \sqrt{x^4 + (y - y^2)^2}\,dx$. Claim: $I'(y) \geq 0$ with equality only in the (clearly non-optimal) case $y = 0$.

To see this, observe that

$$I'(y) = \sqrt{y^4 + (y - y^2)^2} + \int_0^y \frac{(y - y^2)(1 - 2y)}{\sqrt{x^4 + (y - y^2)^2}}\,dx.$$

If $0 < y \leq 1/2$ clearly $I'(y)$ is positive. So suppose $y > 1/2$. Then $I'(y) > 0$ is equivalent to

$$\sqrt{y^4 + (y - y^2)^2} > (y - y^2)(2y - 1)\int_0^y \frac{dx}{\sqrt{x^4 + (y - y^2)^2}}.$$

Since

$$\int_0^y \frac{dx}{\sqrt{x^4 + (y - y^2)^2}} \leq \int_0^y \frac{dx}{\sqrt{(y - y^2)^2}} = \frac{y}{y - y^2},$$

it suffices to show $\sqrt{y^4 + (y - y^2)^2} \geq (2y - 1)y$, $1/2 \leq y \leq 1$. This is the same as

$$\begin{aligned} y^4 + (y - y^2)^2 &\geq (2y - 1)^2 y^2 \\ \Leftrightarrow y^2 + (1 - y)^2 &\geq (2y - 1)^2 \\ \Leftrightarrow 2y^2 - 2y + 1 &\geq 4y^2 - 4y + 1 \\ \Leftrightarrow 2y &\geq 2y^2, \end{aligned}$$

the last of which is clearly true.

Now, for $y < 1$, $I(y) < I(1) = \int_0^1 x^2\,dx = 1/3$, so $1/3$ is the maximum.

Note: If $y_0 < 1/2$ it is easy to see that $I(y_0) < I(1 - y_0)$ since the integrand is nonnegative and $(y(1 - y))^2$ is invariant under $y \to 1 - y$. Hence one may restrict attention to $y \geq 1/2$ from the very beginning.

A-6 (8, 21, 8, 1, 0, 0, 0, 0, 6, 7, 40, 122)

Solution. The sums represented by $A(n)$ may be given an "array" representation using Fibonacci numbers.

Start with a_{r-1} and a_r using two rows of 1's, the lower row with a_r ones and the upper with a_{r-1} ones:

$$a_{r-1}\colon 1\,1\,1\,1\,1\,1\,1$$
$$a_r\colon 1\,1\,1\,1\,1$$

The top row exceeds the bottom row since $a_{r-1} > a_r$.

Now $a_{r-2} > a_{r-1} + a_r$, hence we can uniquely write

$$a_{r-2}\colon 2\,2\,2\,2\,2\,1\,1\,1\,1\,1$$
$$a_{r-1}\colon 1\,1\,1\,1\,1\,1\,1$$
$$a_r\colon 1\,1\,1\,1\,1$$

Next, $a_{r-3} > a_{r-2} + a_{r-1}$, so

$$a_{r-3}\colon 3\,3\,3\,3\,3\,2\,2\,1\,1\,1\,1$$
$$a_{r-2}\colon 2\,2\,2\,2\,2\,1\,1\,1\,1\,1$$
$$a_{r-1}\colon 1\,1\,1\,1\,1\,1\,1$$
$$a_r\colon 1\,1\,1\,1\,1$$

The total array of the representation will involve columns of the form $F_1 + F_2 + F_3 + \cdots + F_s$ and it is easy to see that this is just g_s. That is, by reading columns we see that we have a one-to-one correspondence between the partitions enumerated by $A(n)$ and those enumerated by $B(n)$.

Hence $A(n) = B(n)$ for all n.

B-1 (192, 6, 2, 0, 6, 0, 0, 0, 0, 5, 0, 2)

Solution. If A is a perfect square, the sequence is eventually constant, since it is identically A. Clearly the sequence diverges to infinity if it never contains a perfect square. So, say a_n is not a perfect square, but $a_{n+1} = (r + 1)^2$. If $a_n \geq r^2$ then

$$a_{n+1} = a_n + S(a_n),$$
$$(r + 1)^2 = a_n + (a_n - r^2),$$
$$r^2 + (r + 1)^2 = 2a_n,$$

a contradiction because the left side is odd but the right side is even. On the other hand, if $a_n < r^2$ we have

$$(r + 1)^2 = a_n + S(a_n) < r^2 + \left(r^2 - 1 - (r - 1)^2\right) = r^2 + 2r - 2,$$

again a contradiction. Hence if A is not a perfect square, no a_n is a perfect square.

B-2 (93, 30, 8, 0, 0, 0, 0, 0, 7, 1, 57, 17)

Solution. Differentiate both sides of the two equations with respect to y, obtaining

$$f'(x+y)=f(x)f'(y)-g(x)g'(y),$$
$$g'(x+y)=f(x)g'(y)+g(x)f'(y).$$

Setting $y=0$ yields

$$f'(x)=-g'(0)g(x) \quad \text{and} \quad g'(x)=g'(0)f(x).$$

Thus

$$2f(x)f'(x)+2g(x)g'(x)=0,$$

and therefore

$$(f(x))^2+(g(x))^2=C$$

for some constant C. Since f and g are nonconstant, $C \neq 0$. From the identity

$$[f(x+y)]^2+[g(x+y)]^2=\left[(f(x))^2+(g(x))^2\right]\left[(f(y))^2+(g(y))^2\right],$$

we see that $C=C^2$. Since $C \neq 0$, we have $C=1$.

B-3 (38, 11, 4, 0, 0, 0, 0, 0, 5, 7, 49, 99)

Solution. Yes.

Claim: If a and b are positive integers, then there exists a number L_0 so that every multiple of (a,b) (the greatest common divisor of a and b) greater than L_0 may be written in the form $ra+sb$, where r and s are nonnegative integers.

Proof of Claim: Suppose first that $(a,b)=1$. Then $0, a, 2a, \ldots, (b-1)a$ is a complete set of residues modulo b. Thus, for any integer k greater than $(b-1)a-1$, $k-qb=ja$ for some $q \geq 0$, $j=0,1,2,\ldots,b-1$, hence the claim for this special case.

In general, since $a/(a,b)$ and $b/(a,b)$ are relatively prime, we make use of the above to see that for some L_1, every integer greater than L_1 can be written in the form $ra/(a,b)+sb/(a,b)$. Multiplying through by (a,b) yields the claim.

To answer the question, we begin by forming 20×6 and 20×7 rectangles. From the claim, we may form $20 \times n$ rectangles for n sufficiently large. We may also form 35×5 and 35×7 rectangles, hence $35 \times n$ rectangles for n sufficiently large. We may further form 42×4 and 42×5 rectangles, hence $42 \times n$ rectangles for n sufficiently large.

Since $(20,35)=5$, there exists a multiple m_0 of 5, relatively prime to 42 and independent of sufficiently large n, for which we may form an $m_0 \times n$ rectangle. Finally, since $(m_0, 42)=1$, we may form all $m \times n$ rectangles for m and n sufficiently large.

B-4 (21, 1, 7, 0, 0, 0, 0, 0, 23, 1, 37, 123)

Solution 1. The left side is equal to $\sum_{j=0}^{p}\binom{p}{j}\binom{p+j}{p}$. This is equal to the coefficient of x^p in $((1+x)+1)^p(1+x)^p$. To see this, note that for each j, $\binom{p}{j}$ is the coefficient of $(1+x)^j$ from the first factor, and therefore $\binom{p}{j}\binom{p+j}{p}$ is the coefficient of x^p in $(1+x)^{p+j}$. Summing over j establishes the claim.

On the other hand, the coefficient of x^p in $(2+x)^p(1+x)^p$ is $\sum_{k=0}^{p}\binom{p}{k}\binom{p}{p-k}2^k$. But p divides $\binom{p}{k}$ for $k \neq 0, p$. Thus,

$$\sum_{j=0}^{p}\binom{p}{j}\binom{p+j}{j} = \sum_{j=0}^{p}\binom{p}{j}\binom{p+j}{p} \equiv \binom{p}{0}\binom{p}{p}2^0 + \binom{p}{p}\binom{p}{0}2^p$$
$$\equiv 1 + 2^p (\text{mod } p^2).$$

Solution 2. By the Vandermonde convolution,

$$\sum_{j=0}^{p}\binom{p}{j}\binom{p+j}{j} = \sum_{j\geq 0}\binom{p}{j}\sum_{h\geq 0}\binom{j}{h}\binom{p}{p-h}$$
$$= \sum_{h\geq 0}\binom{p}{p-h}\frac{p!}{h!(p-h)!}\sum_{j\geq 0}\frac{(p-h)!}{(p-j)!(j-h)!}$$
$$= \sum_{h\geq 0}\binom{p}{h}^2 2^{p-h}$$
$$\equiv 2^p + 1 (\text{mod } p^2)$$

since the prime p divides $\binom{p}{h}$ for $0 < h < p$.

B-5 (38, 4, 3, 0, 3, 0, 1, 0, 9, 3, 50, 102)

Solution. There are $\lfloor (p+3)/4 \rfloor$ elements in the intersection.

Consider first the set of solutions to

$$x^2 = y^2 + 1. \qquad (*)$$

Rewriting this as $(x+y)(x-y) = 1$, we see that for each nonzero element r of $\mathbf{Z}_p$, there is exactly one solution to the above, namely, $x + y = r$, $x - y = r^{-1}$, or

$$x = \left(\frac{p+1}{2}\right)(r + r^{-1}), \qquad y = \left(\frac{p+1}{2}\right)(r - r^{-1}).$$

Thus, there are $p - 1$ solutions to $(*)$.

On the other hand, the element $x^2 = y^2 + 1$ in the intersection also arises from the pairs $(x, -y)$, $(-x, y)$, and $(-x, -y)$ as well as (x, y). These four pairs are distinct unless $x = 0$ or $y = 0$, in which case there are just two distinct pairs. Note that 1 arises from $(1, 0)$ and from $(-1, 0)$. Let $c = 1$ if there is a solution with $x = 0$ and let $c = 0$ if not. Then the intersection has $1 + c + d$ elements, where, from the above, $p - 1 = 2 + 2c + 4d$.

We see that $c = 1$ if and only if $p - 1$ is divisible by 4. Solving for d in each case, we find that $1 + c + d = \lfloor (p+3)/4 \rfloor$.

Note: *Ian Richards*, *University of Minnesota*, points out that this problem is a special case ($k = 1$) of the following: If χ is the quadratic character mod p, then

$\sum_{n=0}^{p-1}\chi(n)\chi(n+k) = -1$, independent of k. This follows from the theory of Jacobi or Gauss sums.

B-6 (2, 0, 0, 0, 1, 0, 1, 2, 0, 4, 30, 173)

Solution. The inequality is satisfied if and only if $0 < |u| \leq |\ln(a/b)|$.

The right-hand side is an even function of u; hence it suffices to consider $u > 0$. Replacing x by $1 - x$ and interchanging a and b preserves the inequality, hence we may assume $a \geq b$. Set

$$F(u) = a\frac{\sinh ux}{\sinh u} + b\frac{\sinh u(1-x)}{\sinh u} - a^x b^{1-}$$

By differentiating

$$f(u) = \frac{\sinh ux}{\sinh u}$$

we find that $f'(u) < 0$ if and only if $g(u) = x \tanh u - \tanh xu < 0$. This latter inequality holds because $g(0) = 0$ and $g'(u) < 0$ for $u > 0$. Thus $f(u)$ is strictly decreasing in u, and therefore, so is $F(u)$. If $a > b$ then $F(\ln(a/b)) = 0$, whereas if $a = b$ then $\lim_{u \to 0^+} F(u) = 0$, and the proof is complete.

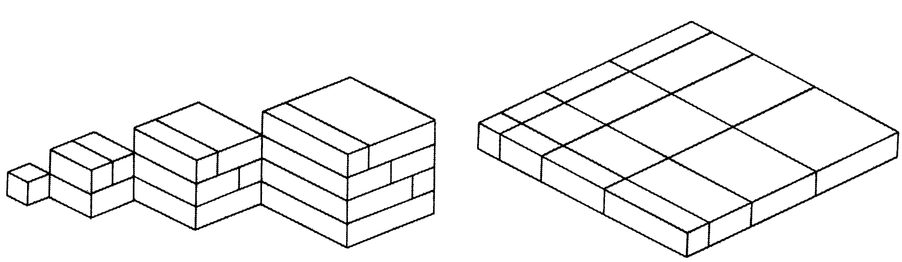
Note: By taking the limit as $u \to 0$, we obtain a proof of the weighted version of the arithmetic-mean–geometric-mean inequality.

Klosinski:
Department of Mathematics
Santa Clara University
Santa Clara, CA 95053

Alexanderson:
Department of Mathematics
Santa Clara University
Santa Clara, CA 95053

Larson:
Department of Mathematics
St. Olaf College
Northfield, MN 55057

BEHOLD



AUTHOR: Michael Hirschhorn