The Fifty-Third William Lowell Putnam Mathematical Competition

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The following results of the fifty-third William Lowell Putnam Mathematical Competition, held on December 5, 1992, have been determined in accordance with the governing regulations. This annual contest is supported by the William Lowell Putnam Prize Fund for the Promotion of Scholarship, left by Mrs. Putnam in memory of her husband, and is held under the auspices of the Mathematical Association of America.

The first prize, $7,500, was awarded to the Department of Mathematics of Harvard University. The members of the winning team were: Jordan S. Ellenberg, Samuel A. Kutin, and Royce Y. Peng; each was awarded a prize of $500.

The second prize, $5,000, was awarded to the Department of Mathematics of the University of Toronto. The members of the winning team were: J. P. Grossman, Jeff T. Higham, and Hugh R. Thomas; each was awarded a prize of $400.

The third prize, $3,000, was awarded to the Department of Mathematics of the University of Waterloo. The members of the winning team were Dorian Birsan, Daniel R. L. Brown, and Ian A. Goldberg; each was awarded a prize of $300.

The fourth prize, $2,000, was awarded to the Department of Mathematics at Princeton University. The members of the winning team were Joshua B. Fischman, Adam M. Logan, and Joel E. Rosenberg; each was awarded a prize of $200.

The fifth prize, $1,000, was awarded to the Department of Mathematics at Cornell University. The members of the winning team were Jon M. Kleinberg, Mark Krosky, and Demetrio A. Muñoz; each was awarded a prize of $100.

The five highest ranking individual contestants, in alphabetical order, were Jordan S. Ellenberg, Harvard University; Samuel A. Kutin, Harvard University; Adam M. Logan, Princeton University; Serban M. Nacu, Harvard University; and Jeffrey M. Vanderkam, Duke University. Each of these was designated a Putnam Fellow by the Mathematical Association of America and awarded a prize of $1,000 by the Putnam Prize Fund.

The next six highest ranking contestants, in alphabetical order, were David B. Carlton, Harvard University; Ian A. Goldberg, University of Waterloo; Kiran S. Kedlaya, Harvard University; Royce Y. Peng, Harvard University; Hugh R. Thomas, University of Toronto; and Tong Zhang, Cornell University; each was awarded a prize of $500.

The next four highest ranking individuals, in alphabetical order, were Ze-Yu Chen, Princeton University; Jonathan T. Higa, Princeton University; Svetlozar E. Nestorov, Stanford University; and Samuel K. Vandervelde, Swarthmore College; each was awarded a prize of $250.

The next nine highest ranking individuals, in alphabetical order, were Daniel R. L. Brown, University of Waterloo; Jeff T. Higham, University of Toronto; F. Dean Hildebrandt, Harvard University; Julie B. Kerr, Washington State University; Andrew H. Kresch, Yale University; William R. Mann, Princeton University; Dana Pascovici, Dartmouth College; Michail G. Sunitsky, Princeton University; and Douglas J. Zare, New College of the University of South Florida; each was awarded a prize of $100.

The following teams, named in alphabetical order, received honorable mention: Dartmouth College, with team members Radu Bacioiu, Rolf H. Nelson, and Dana Pascovici; Duke University, with team members Craig B. Gentry, Alexander J. Hartemink, and Jeffrey M. Vanderkam; Massachusetts Institute of Technology, with team members Thomas C. Chou, Henry L. Cohn, and Michael J. Lawler; University of British Columbia, with team members Malik H. Kalfane, David L. Savitt, and Mark A. Van Raamsdonk; and Yale University, with team members Thomas Feng, Andrew H. Kresch, and Zhaohui Zhang.

Honorable mention was achieved by the following thirty-one individuals named in alphabetical order: James McCleery Berger, Brown University; Sergey Brin, University of Maryland, College Park; Thomas C. Chou, Massachusetts Institute of Technology; Henry L. Cohn, Massachusetts Institute of Technology; Brian D. Ewald, University of Michigan, Ann Arbor; Joshua B. Fischman, Princeton University; J. P. Grossman, University of Toronto; Steven S. Gubser, Princeton University; William M. Hesse, University of Connecticut; Adam Kalai, Harvard University; Timothy P. Kokesh, Harvey Mudd College; Botond Kőszegi, Harvard University; Peter R. Kramer, Princeton University; Mark Krosky, Cornell University; Tal N. Kubo, Harvard University; Sergey V. Levin, Harvard University; Samuel J. Maltby, University of Calgary; Demetrio A. Muñoz, Cornell University; Akira Negi, University of North Carolina, Chapel Hill; Seth Padowitz, Brown University; Andrew Przeworski, Massachusetts Institute of Technology; Philip T. Reiss, University of Manitoba; James P. Sarvis, Massachusetts Institute of Technology; Kannan Soundararajan, University of Michigan, Ann Arbor; Michael G. Szydlo, Boston University; Joe Y. Tien, University of California, Irvine; Mark A. Van Raamsdonk, University of British Columbia; Jeffrey D. Wall, Princeton University; Kelly Lynne Wieand, University of Wisconsin, Madison; Erick B. Wong, Simon Fraser University; and Zhaohui Zhang, Yale University.

The other individuals who achieved ranks among the top 98, in alphabetical order of their schools, were: Brigham Young University, John Wesley Robertson; University of British Columbia, David L. Savitt; Brown University, Andrew Brecher; California Institute of Technology, Steven C. Anderson; University of California, Berkeley, Daniel C. Isaksen; University of Colorado, Boulder, Steve T. Soulé; Cornell University, Jon M. Kleinberg; Dartmouth College, Radu Bacioiu; Duke University, Alexander J. Hartemink; Harvard University, Manjul Bhargava, Joseph I. Chuang, Michael L. Hutchings, Dimitri Kountourogiannis, Paul Li, Matteo J. Paris, Chris Ternoey; Harvey Mudd College, Jon H. Leonard; University of Maine, Orono, YuQun Chen; Massachusetts Institute of Technology, Jerome S. Khohayting, Tichomir G. Tenev, William W. Tucker; Memorial University of Newfoundland, Robert P. Gallant; Michigan State University, Thomas P. Hayes; University of Minnesota, Minneapolis, Matthew P. Kelly; Université de Montréal, Marc-André

Lafortune; New York University, Mikhail Kogan; Ohio State University, Frank J. Swenton; University of Pennsylvania, Frosti Petursson; Princeton University, Tibor Beke, Mark W. Lucianovic; Purdue University, Pok-Yin Yu; Rice University, Donald A. Barkauskas; Rose Hulman Institute of Technology, Jonathan E. Atkins; Stanford University, Daniel P. Cory, Garrett R. Vargas; Texas A & M University, Zheng-Zheng Li; University of Waterloo, Dorian Birsan, Kevin K. Cheung, Jie J. Lou; Wellesley College, Yihao L. Zhang; West Virginia Wesleyan College, Emanuel V. Todorov; and Yale University, Matthew Frank.

The Elizabeth Lowell Putnam Prize, named for the wife of William Lowell Putnam and to be "awarded periodically to a woman whose performance on the Competition has been deemed particularly meritorious", is awarded this year for the first time to Dana Pascovici of Dartmouth College. The winner is awarded a prize of $500.

There were 2421 individual contestants from 393 colleges and universities in Canada and the United States in the competition of December 5, 1992. Teams were entered by 284 institutions.

The Questions Committee for the fifty-third competition consisted of George E. Andrews (Chair), George T. Gilbert, and Eugene Luks; they composed the problems listed below and were most prominent among those suggesting solutions.

PROBLEMS

Problem A-1.

Prove that $f(n) = 1 - n$ is the only integer-valued function defined on the integers that satisfies the following conditions:

(i) $f(f(n)) = n$, for all integers n;
(ii) $f(f(n+2)+2) = n$ for all integers n;
(iii) $f(0) = 1$.

Problem A-2.

Define $C(\alpha)$ to be the coefficient of x^{1992} in the power series expansion about $x = 0$ of $(1+x)^\alpha$. Evaluate

$$\int_0^1 C(-y-1)\left(\frac{1}{y+1} + \frac{1}{y+2} + \frac{1}{y+3} + \cdots + \frac{1}{y+1992}\right) dy.$$

Problem A-3.

For a given positive integer m, find all triples (n, x, y) of positive integers, with n relatively prime to m, which satisfy $(x^2 + y^2)^m = (xy)^n$.

Problem A-4.

Let f be an infinitely differentiable real-valued function defined on the real numbers. If

$$f\left(\frac{1}{n}\right) = \frac{n^2}{n^2+1}, \qquad n = 1, 2, 3, \ldots,$$

compute the values of the derivatives $f^{(k)}(0)$, $k = 1, 2, 3, \ldots$.

Problem A-5.

For each positive integer n, let

$$a_n = \begin{cases} 0 & \text{if the number of 1's in the binary representation of } n \text{ is even,} \\ 1 & \text{if the number of 1's in the binary representation of } n \text{ is odd.} \end{cases}$$

Show that there do not exist positive integers k and m such that

$$a_{k+j} = a_{k+m+j} = a_{k+2m+j}, \quad \text{for } 0 \le j \le m-1.$$

Problem A-6.

Four points are chosen at random on the surface of a sphere. What is the probability that the center of the sphere lies inside the tetrahedron whose vertices are at the four points? (It is understood that each point is independently chosen relative to a uniform distribution on the sphere.)

Problem B-1.

Let S be a set of n distinct real numbers. Let A_S be the set of numbers that occur as averages of two distinct elements of S. For a given $n \ge 2$, what is the smallest possible number of distinct elements in A_S?

Problem B-2.

For nonnegative integers n and k, define $Q(n, k)$ to be the coefficient of x^k in the expansion of $(1 + x + x^2 + x^3)^n$. Prove that

$$Q(n,k) = \sum_{j=0}^{n} \binom{n}{j}\binom{n}{k-2j},$$

where $\binom{a}{b}$ is the standard binomial coefficient. (Reminder: For integers a and b with $a \ge 0$, $\binom{a}{b} = a!/(b!(a-b)!)$ for $0 \le b \le a$, and $\binom{a}{b} = 0$ otherwise.)

Problem B-3.

For any pair (x, y) of real numbers, a sequence $(a_n(x, y))_{n \ge 0}$ is defined as follows:

$$a_0(x,y) = x,$$

$$a_{n+1}(x,y) = \frac{(a_n(x,y))^2 + y^2}{2}, \quad \text{for all } n \ge 0.$$

Find the area of the region $\{(x, y) \mid (a_n(x, y))_{n \ge 0} \text{ converges}\}$.

Problem B-4.

Let $p(x)$ be a nonzero polynomial of degree less than 1992 having no nonconstant factor in common with $x^3 - x$. Let

$$\frac{d^{1992}}{dx^{1992}}\left(\frac{p(x)}{x^3 - x}\right) = \frac{f(x)}{g(x)}$$

for polynomials $f(x)$ and $g(x)$. Find the smallest possible degree of $f(x)$.

Problem B-5.

Let D_n denote the value of the $(n-1) \times (n-1)$ determinant

$$\begin{vmatrix} 3 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 4 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 5 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 6 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & n+1 \end{vmatrix}.$$

Is the set $\{D_n/n!\}_{n \geq 2}$ bounded?

Problem B-6.

Let $\mathcal{M}$ be a set of real $n \times n$ matrices such that

(i) $I \in \mathcal{M}$, where I is the $n \times n$ identity matrix;
(ii) if $A \in \mathcal{M}$ and $B \in \mathcal{M}$, then either $AB \in \mathcal{M}$ or $-AB \in \mathcal{M}$, but not both;
(iii) if $A \in \mathcal{M}$ and $B \in \mathcal{M}$, then either $AB = BA$ or $AB = -BA$;
(iv) if $A \in \mathcal{M}$ and $A \neq I$, there is at least one $B \in \mathcal{M}$ such that $AB = -BA$.

Prove that $\mathcal{M}$ contains at most n^2 matrices.

SOLUTIONS

In the 12-tuples $(n_{10}, n_9, \ldots, n_0, n_{-1})$ following each problem number below, n_i for $10 \geq i \geq 0$ is the number of students among the top 203 contestants achieving i points for the problem and n_{-1} is the number of those not submitting solutions.

A − 1 (31, 82, 42, 10, 0, 0, 0, 7, 23, 6, 2, 0)

Solution. If $f(n) = 1 - n$, then $f(f(n)) = f(1-n) = 1 - (1-n) = n$, so (i) holds. Similarly, $f(f(n+2)+2) = f((-n-1)+2) = f(1-n) = n$, so (ii) holds. Clearly (iii) holds, and so $f(n) = 1 - n$ satisfies the conditions.

Conversely, suppose f satisfies the three given conditions. From condition (ii), $f(f(f(n+2)+2)) = f(n)$, and applying (i) yields $f(n+2)+2 = f(n)$ or $f(n+$

2) $= f(n) - 2$. An easy induction yields

$$f(n) = \begin{cases} f(0) - n & \text{if } n \text{ is even,} \\ f(1) + 1 - n & \text{if } n \text{ is odd.} \end{cases}$$

If $f(0) = 1$, then $f(1) = 0$ by (i), therefore, $f(n) = 1 - n$.

A-2 (157, 1, 0, 0, 0, 0, 0, 0, 2, 14, 14, 15)

Solution. From the binomial series, we see that

$$C(-y-1) = \frac{(-y-1)(-y-2)\cdots(-y-1992)}{1992!} = \frac{(y+1)(y+2)\cdots(y+1992)}{1992!}.$$

Therefore,

$$C(-y-1)\left(\frac{1}{y+1} + \frac{1}{y+2} + \cdots + \frac{1}{y+1992}\right) = \frac{d}{dy}\left(\frac{(y+1)(y+2)\cdots(y+1992)}{1992!}\right).$$

Hence the integral in question is

$$\int_0^1 \frac{d}{dy}\left(\frac{(y+1)(y+2)\cdots(y+1992)}{1992!}\right) dy = \left.\frac{(y+1)(y+2)\cdots(y+1992)}{1992!}\right|_0^1 = 1993 - 1 = 1992.$$

A-3 (55, 20, 7, 0, 0, 0, 0, 0, 16, 7, 45, 53)

Solution. There are no solutions if m is odd. If m is even, the only solution is $(n, x, y) = (m + 1, 2^{m/2}, 2^{m/2})$.

If (n, x, y) is a solution, then by the arithmetic-mean—geometric-mean inequality, $(xy)^n = (x^2 + y^2)^m \geq (2xy)^m$, so $n > m$. Let p be a prime number. Let a and b be the largest powers of p that divide x and y, respectively. Then the largest power of p dividing $(xy)^n$ is $(a + b)n$. If $a < b$, the largest power of p dividing $(x^2 + y^2)^m$ is $2am$. But this implies that $(a + b)n = 2am$, and this contradicts $n > m$. Similarly, the assumption $a > b$ leads to a contradiction. Therefore $a = b$ for all primes p, and we conclude that $x = y$. Thus, the equation reduces to $(2x^2)^m = x^{2n}$, or equivalently, $x^{2(n-m)} = 2^m$. It follows that x is a positive power of 2, say 2^a. This implies $2(n - m)a = m$, or, $2an = (2a + 1)m$. Since $\gcd(m, n) = \gcd(2a, 2a + 1) = 1$, we must have $m = 2a$ and $n = 2a + 1$. Thus, m is necessarily even and the solution follows as claimed.

A-4 (17, 6, 7, 0, 0, 0, 2, 0, 73, 18, 47, 33)

Solution. We will show that

$$f^{(k)}(0) = \begin{cases} (-1)^{k/2} k! & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

First we note that if $h(x)$ is a differentiable function and $x_1, x_2, \ldots,$ is a sequence strictly decreasing to 0 such that $h(x_n) = 0$, then by Rolle's Theorem, there exists a sequence $y_1, y_2, \ldots,$ strictly decreasing to 0, such that $h'(y_n) = 0$ $(x_{n+1} < y_n < x_n)$.

Now let $g(x) = f(x) - 1/(1 + x^2)$. Then $g(1/n) = 0$ for $n = 1, 2, \ldots$. Applying the result of the preceding paragraph to $g, g', g'', \ldots$ and invoking the continuity of $g^{(k)}$ at 0, we see that $g^{(k)}(0) = 0$ for $k = 0, 1, 2, 3, \ldots$. Thus,

$$f^{(k)}(0) = \frac{d^k}{dx^k}\left(\frac{1}{1+x^2}\right)\bigg|_{x=0}.$$

The Maclaurin series for $1/(1 + x^2)$ is $\sum_{k=0}^{\infty}(-1)^k x^{2k}$, and hence $f^{(k)}(0)$ is equal to the values given above.

A-5 (1, 9, 1, 0, 0, 0, 0, 0, 5, 3, 72, 112)

Solution. Observe that $a_{2n} = a_n$ and $a_{2n+1} = 1 - a_{2n} = 1 - a_n$.

Suppose that there exist k, m as above, and we may assume m is minimal for such.

Suppose first that m is odd. We'll suppose $a_k = a_{k+m} = a_{k+2m} = 0$, as it will be clear that the case $a_k = 1$ can be treated similarly. Since either k or $k + m$ is even, $a_{k+1} = a_{k+m+1} = a_{k+2m+1} = 1$. Again, since either $k + 1$ or $k + m + 1$ is even, $a_{k+2} = a_{k+m+2} = a_{k+2m+2} = 0$. By this means, we see that the terms $a_k, a_{k+1}, a_{k+2}, \ldots, a_{k+m-1}$ alternate between 0 and 1. Then since $m - 1$ is even, $a_{k+m-1} = a_{k+2m-1} = a_{k+3m-1} = 0$. But, since either $k + m - 1$ or $k + 2m - 1$ is even, that would imply that $a_{k+m} = a_{k+2m} = 1$, a contradiction.

Thus, m must be even. Extracting the terms with even indices in

$$a_{k+j} = a_{k+m+j} = a_{k+2m+j}, \quad \text{for } 0 \le j \le m - 1,$$

and using the fact that $a_r = a_{r/2}$ for even r, we get

$$a_{\lceil k/2\rceil+i} = a_{\lceil k/2\rceil+(m/2)+i} = a_{\lceil k/2\rceil+m+i}, \quad \text{for } 0 \le i \le (m/2) - 1.$$

(The even numbers $\ge k$ are $2\lceil k/2\rceil, 2\lceil k/2\rceil + 2, \ldots$.) This contradicts the minimality of m.

Hence, there are no such k and m.

A-6 (9, 3, 4, 0, 0, 0, 0, 0, 0, 10, 32, 22, 123)

Solution. Recall first that if points A, B, C, D are in general position in 3-space, then a point E lies inside the tetrahedron $ABCD$ if and only if the barycentric coordinates of E with respect to A, B, C, D are positive. That is, if we (uniquely) express

$$\vec{E} = w\vec{A} + x\vec{B} + y\vec{C} + z\vec{D}, \quad \text{with } w + x + y + z = 1,$$

(the arrows indicating consideration of the coordinate triples as vectors), then E is in the interior of $ABCD$ if and only if $w > 0$, $x > 0$, $y > 0$, and $z > 0$. Hence, if E is the origin, then E is in the interior of $ABCD$ if and only if there is a solution (w, x, y, z) to

$$\vec{0} = w\vec{A} + x\vec{B} + y\vec{C} + z\vec{D} \tag{1}$$

with w, x, y, z having the same sign. As the solution space to (1) is 1-dimensional, this condition holds for one nonzero solution if and only if it holds for all.

Now assume that the center of the sphere is located at the origin and fix the first chosen point P on the sphere as the north pole, the other three points, P_1, P_2, P_3, then being random.

We may suppose the choice of each P_i is made in two steps, the first choosing a random diameter $Q_{i_1}Q_{i_2}$ and the second choosing at random between the endpoints Q_{i_1}, Q_{i_2}. Since the $2^3 = 8$ possible selections of endpoints of the three diameters are equally likely, each of the 8 tetrahedra $PQ_{1j_1}Q_{2j_2}Q_{3j_3}$, $j_i = 1$ or 2, are equally likely. We may further suppose that the vertices of each of these tetrahedra are in general position as the probability of degeneracy is 0. Similarly, we may suppose that the center of the sphere does not lie on any face of the tetrahedra.

Let (w, x, y, z) be a nonzero solution to the equation

$$\vec{0} = w\vec{P} + x\vec{Q}_{11} + y\vec{Q}_{21} + z\vec{Q}_{31}.$$

Then, since $\vec{Q}_{i1} = -\vec{Q}_{i2}$, the eight equations

$$\vec{0} = w\vec{P} + x\vec{Q}_{1j_1} + y\vec{Q}_{2j_2} + z\vec{Q}_{3j_3}$$

have respective solutions

$$(w, x, y, z), (w, x, y, -z), (w, x, -y, z), (w, -x, y, z),$$

$$(w, x, -y, -z), (w, -x, -y, z), (w, -x, y, -z), (w, -x, -y, -z).$$

Hence, exactly one of the eight equations has a solution whose coordinates have the same sign.

It follows that exactly one of these 8 equally likely tetrahedra contains the center. Thus the probability of including the center is $1/8$ for all initial choices of 3 diameters. We conclude that the probability for a random tetrahedron is $1/8$.

B-1 (145, 15, 4, 0, 0, 0, 0, 0, 0, 6, 14, 11, 8)

Solution. The smallest possible number of elements in A_S is $2n - 3$.

Let $x_1 < x_2 < \cdots < x_n$ represent the elements of S. Then

$$\frac{x_1 + x_2}{2} < \frac{x_1 + x_3}{2} < \cdots < \frac{x_1 + x_n}{2} < \frac{x_2 + x_n}{2} < \frac{x_3 + x_n}{2}$$
$$< \cdots < \frac{x_{n-1} + x_n}{2}$$

represent $(n - 1) + (n - 2) = 2n - 3$ distinct elements of A_S, so A_S has at least $2n - 3$ distinct elements.

On the other hand, if we take $S = \{1, 2, \ldots, n\}$, the elements of A_S are $\frac{3}{2}, \frac{4}{2}, \frac{5}{2}, \ldots, \frac{2n-1}{2}$. There are only $(2n - 1) - 2 = 2n - 3$ such numbers; thus there is a set A_S with at most $2n - 3$ distinct elements. This completes the proof.

B-2 (159, 10, 7, 0, 0, 0, 0, 0, 1, 4, 13, 9)

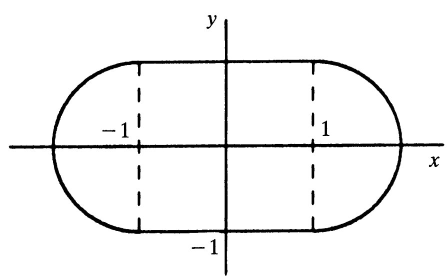
Solution. We have

$$\begin{aligned}\sum_{k\geq 0} Q(n,k)x^k &= (1+x+x^2+x^3)^n \\ &= (1+x^2)^n(1+x)^n \\ &= \sum_{j\geq 0}\binom{n}{j}x^{2j}\sum_{i\geq 0}\binom{n}{i}x^i \\ &= \sum_{j\geq 0}\sum_{i\geq o} x^{2j+i}\binom{n}{j}\binom{n}{i} \\ &= \sum_{k\geq 0} x^k\sum_{j\geq 0}\binom{n}{j}\binom{n}{k-2j}.\end{aligned}$$

Comparing coefficients of x^k, we derive the desired result.

B-3 (23, 11, 10, 0, 0, 0, 0, 0, 27, 24, 71, 37)

Solution. The area is $4+\pi$. The region of convergence is



namely, a (closed) square $\{(x,y)| -1 \leq x, y \leq 1\}$ of side 2 with (closed) semicircles of radius 1 centered at $(\pm 1, 0)$ described on two opposite sides.

If $\lim_{n\to\infty} a_n(x,y) = L$, then L must satisfy $L = (L^2+y^2)/2$; that is, L must be a root of the equation

$$r^2 - 2r + y^2 = 0. \tag{1}$$

In such case, the equation must have real roots, so the discriminant, $4-4y^2$, must be nonnegative. Thus, a necessary condition for $(a_n(x,y))$ to converge is that $|y| \leq 1$.

Fix $|y| \leq 1$. The roots of (1) are then $1-\sqrt{1-y^2}$ and $1+\sqrt{1-y^2}$, which are real and nonnegative. As $a_1(-x,y) = a_1(x,y)$, the interval of convergence is symmetric about $x=0$. We shall assume then that $x \geq 0$; thus, $a_n(x,y) \geq 0$, for all n.

If $r_0 = 1 \pm \sqrt{1-y^2}$, then $a_{n+1}(x,y)$ is less than, equal to, or greater than r_0 according to whether $a_n(x,y)$ is less than, equal to, or greater than r_0 $(= (r_0^2+y^2)/2)$.

If $a_n(x, y)$ lies in the closed interval $[1 - \sqrt{1-y^2}, 1 + \sqrt{1-y^2}]$, that is, between the roots of (1), then

$$a_n(x,y)^2 - 2a_n(x,y) + y^2 \le 0,$$

so that

$$1 - \sqrt{1-y^2} \le a_{n+1}(x,y) \le a_n(x,y).$$

It follows that $(a_n(x, y))_{n \ge 0}$ converges if x is in the closed interval $[1 - \sqrt{1-y^2}, 1 + \sqrt{1-y^2}]$.

If $a_n(x, y)$ does not lie in the interval $[1 - \sqrt{1-y^2}, 1 + \sqrt{1-y^2}]$, then

$$a_n(x,y)^2 - 2a_n(x,y) + y^2 > 0,$$

so that

$$a_{n+1}(x,y) > a_n(x,y).$$

Thus, if x, and therefore all $a_n(x, y)$, are greater than $1 + \sqrt{1-y^2}$, then the sequence diverges. On the other hand, if x, and therefore all $a_n(x, y)$, lie between 0 and $1 - \sqrt{1-y^2}$, the sequence converges monotonically to $1 - \sqrt{1-y^2}$.

To summarize, $(a_n(x, y))_{n \ge 0}$ converges if and only if

$$-1 \le y \le 1$$

and

$$-\left(1 + \sqrt{1-y^2}\right) \le x \le 1 + \sqrt{1-y^2}.$$

B-4 (35, 11, 13, 0, 0, 0, 0, 0, 12, 5, 48, 79)

Solution. The smallest possible degree of $f(x)$ is 3984.

By the Division Algorithm, we can write $p(x) = (x^3 - x)q(x) + r(x)$, where $q(x)$ and $r(x)$ are polynomials, the degree of $r(x)$ is less than 3, and the degree of $q(x)$ is less than 1989. Then

$$\frac{d^{1992}}{dx^{1992}}\left(\frac{p(x)}{x^3 - x}\right) = \frac{d^{1992}}{dx^{1992}}\left(\frac{r(x)}{x^3 - x}\right).$$

Now, write $r(x)/(x^3 - x)$ in the form

$$\frac{A}{x-1} + \frac{B}{x} + \frac{C}{x+1}.$$

Because $p(x)$ and $x^3 - x$ have no nonconstant common factor, neither do $r(x)$ and $x^3 - x$, and therefore, $ABC \neq 0$. Thus,

$$\begin{aligned}
&\frac{d^{1992}}{dx^{1992}}\left(\frac{r(x)}{x^3 - x}\right)\\
&\quad = 1992!\left(\frac{A}{(x-1)^{1993}} + \frac{B}{x^{1993}} + \frac{C}{(x+1)^{1993}}\right)\\
&\quad = 1992!\left(\frac{Ax^{1993}(x+1)^{1993} + B(x-1)^{1993}(x+1)^{1993} + C(x-1)^{1993}x^{1993}}{(x^3-x)^{1993}}\right).
\end{aligned}$$

Since $ABC \neq 0$, it is clear that the numerator and denominator have no common factor. Expanding the numerator yields an expression of the form

$$(A+B+C)x^{3986} + 1993(A-C)x^{3985} + 1993(996A - B + 996C)x^{3984} + \cdots .$$

From $A = C = 1$, $B = -2$, we see the degree can be as low as 3984. A lower degree would imply $A + B + C = 0$, $A - C = 0$, $996A - B + 996C = 0$, implying that $A = B = C = 0$, a contradiction.

B-5 (62, 4, 4, 0, 0, 0, 0, 3, 6, 2, 49, 73)

Solution 1. The set $\{D_n/n!\}_{n\geq 2}$ forms a sequence which strictly increases to infinity; it is therefore unbounded.

Observing that $D_2 = 3$ and $D_3 = 11$, we obtain a recursion for D_{n+1}. Subtracting the next-to-last column from the last column and then the next-to-last row from the last row, one finds

$$D_{n+1} = \det\begin{pmatrix} 3 & 1 & 1 & \cdots & & 1 & 0 \\ 1 & 4 & 1 & \cdots & & 1 & 0 \\ 1 & 1 & 5 & \cdots & & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & n+1 & -n \\ 0 & 0 & 0 & \cdots & 0 & -n & 2n+1 \end{pmatrix}.$$

Expanding the determinant in its last row, one obtains

$$D_{n+1} = (2n+1)D_n - n^2 D_{n-1}.$$

Letting $r_n = (D_n/n!)$, the recursion may be written as

$$r_{n+1} = \frac{2n+1}{n+1} r_n - \frac{n}{n+1} r_{n-1},$$

or

$$(r_{n+1} - r_n) = \frac{n}{n+1}(r_n - r_{n-1}).$$

We conclude that

$$r_{n+1} - r_n = \frac{3}{n+1}(r_3 - r_2) = \frac{1}{n+1}.$$

Therefore,

$$\begin{aligned} r_{n+1} &= r_2 + (r_3 - r_2) + (r_4 - r_3) + \cdots + (r_{n+1} - r_n) \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1}, \end{aligned}$$

so the sequence (r_n) diverges to infinity.

Solution 2. The problem is the case $a_i = i + 1$ of

$$D_{n+1}(a_1, \ldots, a_n) = \det\begin{pmatrix} 1+a_1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 1+a_2 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1+a_3 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 1+a_4 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 1+a_n \end{pmatrix}$$

$$= \prod_{i=1}^{n} a_i + \sum_{i=1}^{n} \prod_{\substack{j=1 \\ j \neq i}}^{n} a_j.$$

This formula follows immediately from the recurrence

$$D_{n+1}(a_1, \ldots, a_n) = a_n D_n(a_1, \ldots, a_{n-1}) + a_{n-1} D_n(a_1, \ldots, a_{n-2}, 0).$$

To prove this recurrence, subtract the $(n-1)$st column from the nth column, and then expand along the nth column.

If none of the a_i's equal 0, we can write the polynomial $D_n(a_1, \ldots, a_{n-1})$ in the form

$$D_n(a_1, \ldots, a_{n-1}) = a_1 a_2 \cdots a_{n-1}\left(1 + \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{n-1}}\right).$$

It follows that

$$\frac{D_n}{n!} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n},$$

so the sequence $(D_n/n!)$ is unbounded.

B-6 (0, 0, 0, 0, 0, 0, 0, 0, 0, 5, 4, 39, 155)

Solution 1. We prove the result more generally for complex matrices (because it is convenient to use $i = \sqrt{-1}$ in the proof).

The proof is by induction on n.

If $n = 1$ then the elements of $\mathscr{M}$ commute so that (iv) cannot be satisfied unless $\mathscr{M} = \{I\}$. Suppose that $n > 1$ and that the result holds for sets of complex matrices of smaller dimension.

We may assume $|\mathscr{M}| > 1$, so by (iv), there exist $C, D \in \mathscr{M}$ with $CD = -DC$. Fix such C, D. As in the first solution, $C^2 = \pm I$. Hence the eigenvalues of C are $\pm\lambda$ where $\lambda = 1$ or i. Furthermore, $\mathbf{C}^n = V_\lambda \oplus V_{-\lambda}$, where $V_\lambda, V_{-\lambda}$ are the nullspaces of $(C - \lambda I), (C + \lambda I)$ respectively. We observe that if $X \in \mathscr{M}$ then

$$CX = XC \Rightarrow (C \pm \lambda I)X = X(C \pm \lambda I) \Rightarrow V_{\pm\lambda}X = V_{\pm\lambda};$$

$$CX = -XC \Rightarrow (C \pm \lambda I)X = (-1)X(C \mp \lambda I) \Rightarrow V_{\pm\lambda}X = V_{\mp\lambda}.$$

In particular, since $V_\lambda D = V_{-\lambda}$, $\dim(V_\lambda) = \dim(V_{-\lambda}) = n/2$.

Let $\mathscr{N} = \{X \in \mathscr{M} \mid CX = XC,\ DX = XD\}$. If $Y \in \mathscr{M}$ then exactly one of Y, YC, YD, YCD is in $\mathscr{N}$. It follows that $|\mathscr{N}| = |\mathscr{M}|/4$.

For $X \in \mathscr{N}$, let $\phi(X)$ be the $n/2 \times n/2$ matrix representing, with respect to a fixed basis of V_λ, the linear transformation given by $v \to vX$ for $v \in V_\lambda$. Then ϕ is injective. To see this: assume $\phi(X) = \phi(Y)$ so that $vX = vY$ for $v \in V_\lambda$; but if $v \in V_{-\lambda}$ then $vD \in V_\lambda$, so that $vXD = vDX = vDY = vYD$, which again implies

$vX = vY$; since X, Y induce the same transformations of both V_λ and $V_{-\lambda}$, it follows that $X = Y$.

If suffices finally to show that $\phi(\mathscr{N})$, a set of $n/2 \times n/2$ complex matrices, satisfies (i), (ii), (iii), (iv), for then, by induction, $|\phi(\mathscr{N})| \le (n/2)^2$, whence $|\mathscr{M}| = 4|\mathscr{N}| = 4|\phi(\mathscr{N})| \le n^2$.

Conditions (i), (ii), (iii) for $\phi(\mathscr{N})$ are clearly inherited from those of $\mathscr{M}$. To show (iv), let $\phi(A) \in \phi(\mathscr{N})$, with $\phi(A)$ not the $n/2 \times n/2$ identity matrix. Then $A \neq I$ (as ϕ is injective) and $AB = -BA$ for some $B \in \mathscr{M}$. Let B' be the element of $\{B, BC, BD, BCD\}$ belonging to $\mathscr{N}$. Since $AB' = -B'A$, $\phi(A)\phi(B') = -\phi(B')\phi(A)$.

Solution 2. Let G be the group $\{\pm A \mid A \in \mathscr{M}\}$. We must show that $|G| \le 2n^2$.

The center of $G, Z(G)$, consists of $\pm I$, and if $X \in G \setminus Z(G)$, then X has precisely two conjugates, namely itself and $-X$. Thus G has $1 + |G|/2$ conjugacy classes, and therefore, G has $1 + |G|/2$ inequivalent irreducible representations over $\mathbf{C}$.

The number of inequivalent representations of dimension 1 is $|G/G'|$, where G' is the commutator subgroup. Since $G' = \{\pm I\} = Z(G)$, this number is $|G|/2$.

The remaining irreducible representation then has dimension $\sqrt{|G|/2}$ (since the sum of the squares of the dimensions of the irreducible representations is $|G|$). This representation must be contained in the given representation of G in $n \times n$ matrices, for in all the 1-dimensional representations, $Z(G)$ is in the kernel. Hence $n \ge \sqrt{|G|/2}$, or $2n^2 \ge |G|$.

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Professor H. B. FINE, of Princeton University, was fatally injured by an automobile on the evening of Friday, December 21 and died about one A.M. on December 22, 1928. He was seventy years of age.

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