The Fifty-Fourth William Lowell Putnam Mathematical Competition

Leonard F. Klosinski, Gerald L. Alexanderson, and Loren C. Larson

The following results of the fifty-fourth William Lowell Putnam Mathematical Competition, held on December 4, 1993, have been determined in accordance with the governing regulations. This annual contest is supported by the William Lowell Putnam Prize Fund for the Promotion of Scholarship, left by Mrs. Putnam in memory of her husband, and is held under the auspices of the Mathematical Association of America.

The first prize, $7,500, was awarded to the Department of Mathematics of Duke University. The members of the winning team were: Andrew O. Dittmer, Craig B. Gentry, and Jeffrey M. Vanderkam; each was awarded a prize of $500.

The second prize, $5,000, was awarded to the Department of Mathematics of Harvard University. The members of the winning team were Kiran S. Kedlaya, Serban M. Nacu, and Royce Y. Peng; each was awarded a prize of $400.

The third prize, $3,000, was awarded to the Department of Mathematics of Miami University. The members of the winning team were John D. Davenport, Jason A. Howald, and Matthew D. Wolf; each was awarded a prize of $300.

The fourth prize, $2,000, was awarded to the Department of Mathematics of the Massachusetts Institute of Technology. The members of the winning team were Henry L. Cohn, Alexandru D. Ionescu, and Andrew Przeworski; each was awarded a prize of $200.

The fifth prize, $1,000, was awarded to the Department of Mathematics of the University of Michigan, Ann Arbor. The members of the winning team were Philip L. Beineke, Brian D. Ewald, and Soundararajan Kannan.

The six highest ranking individual contestants, in alphabetical order, were Craig B. Gentry, Duke University; J. P. Grossman, University of Toronto; Wei-Hwa Huang, California Institute of Technology; Kiran S. Kedlaya, Harvard University; Adam M. Logan, Princeton University; and Lenhard L. Ng, Harvard University. Each of these was designated a Putnam Fellow by the Mathematical Association of America and awarded a prize of $1,000, by the Putnam Prize Fund.

The next three highest ranking contestants, in alphabetical order, were Michail Sunitsky, Princeton University; Dylan P. Thurston, Harvard University; and Jade P. Vinson, Washington University, St. Louis; each was awarded a prize of $500.

The next six highest ranking contestants, in alphabetical order, were Mikhail Kogan, New York University; Akira Negi, University of North Carolina, Chapel Hill; Joel E. Rosenberg, Princeton University; David L. Savitt, University of British Columbia; Jeffrey D. Wall, Princeton University; and Thomas A. Weston, Massachusetts Institute of Technology; each was awarded a prize of $250.

The next thirteen highest ranking contestants, in alphabetical order, were Manjul Bhargava, Harvard University; John D. Davenport, Miami University; Andrew O. Dittmer, Duke University; Sergey V. Levin, Harvard University; Douglas A. Levy, University of Pennsylvania; William R. Mann, Princeton University; Adam W. Meyerson, Massachusetts Institute of Technology; Curtis Z. Mitchell, Carleton College; Serban M. Nacu, Harvard University; An T. Nguyen, University of Texas, Austin; Byron M. Shock, Albertson College of Idaho; Ka-Ping R. Yee, University of Waterloo; and Douglas J. Zare, New College of the University of South Florida; each was awarded a prize of $100.

The following teams, named in alphabetical order, received honorable mention: Cornell University, with team members Robert D. Kleinberg, Mark Krosky, and Tong Zhang; New York University, with team members Igor Berger, Yevgeniy Dodis, and Mikhail Kogan; Princeton University, with team members Ze Y. Chen, Adam M. Logan, and William R. Mann; University of Toronto, with team members J. P. Grossman, Edwin N. Sato, and Hugh Thomas; and the University of Waterloo, with team members Daniel R. L. Brown, Ian A. Goldberg, and Peter L. Milley.

Honorable mention was achieved by the following twenty-nine individuals named in alphabetical order: Jared E. Anderson, University of Victoria; Jonathan E. Atkins, Rose-Hulman Institute of Technology; Henry L. Cohn, Massachusetts Institute of Technology; Ilya A. Entin, Massachusetts Institute of Technology; Brian D. Ewald, University of Michigan, Ann Arbor; Kevin E. Foltz, Rice University; David Friedman, Massachusetts Institute of Technology; Ian A. Goldberg, University of Waterloo; H. Tracy Hall, Brigham Young University; John D. Harrington, University of Idaho; Simeon J. Hellerman, Brown University; Timothy J. Hollebeek, Calvin College; Alexandru D. Ionescu, Massachusetts Institute of Technology; Daniel C. Isaksen, University of California, Berkeley; Soundararajan Kannan, University of Michigan, Ann Arbor; Robert D. Kleinberg, Cornell University; Botand Kőszegi, Harvard University; Josh L. Levenberg, Reed College; Jie J. Lou, University of Waterloo; Idris D. Mercer, University of Victoria; Frosti Petursson, University of Pennsylvania; Anand J. Reddy, University of California, Berkeley; Edwin N. Sato, University of Toronto; Jason R. Schweinsberg, Williams College; Mark A. Van Raamsdonk, University of British Columbia; Jeffrey M. Vanderkam, Duke University; Wayne A. Whitney, Harvard University; Tong Zhang, Cornell University; and Zhaohui Zhang, Yale University.

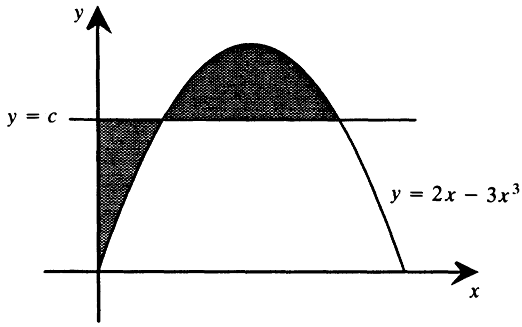
The other individuals who achieved ranks among the top 101, in alphabetical order of their schools, were: University of Arizona, Randy W. Ho; Brigham Young University, John Wesley Robertson; California Institute of Technology, Julian C. Jamison; University of California, San Diego, David R. Wasserman; University of Chicago, Dean W. Jens; Colorado State University, Matthew K. Kahle; Cornell University, Demetrio A. Muñoz; Gordon College, Hai Shao; Harvard University, Swaine L. Chen, Hank S. Chien, Adam Kalai, Joseph D. Kanapka, Joshua N. Newman, David S. Patterson, Royce Y. Peng, Jonathan E. Tannenhauser; Massachusetts Institute of Technology, Andrew Przeworski, Dmitriy A. Rogozhnikov, Jason M. Sachs, Ritesh A. Shah; McGill University, Rajesh J. Pereira; Miami University, Jason A. Howald, Matthew D. Wolf; University of Michigan, Ann Arbor, Philip L. Beineke; University of Minnesota, Minneapolis, Matthew P. Kelly; New York University, Yevgeniy Dodis; North Carolina State University, Stephen A. London; University of Northern Colorado, John C. Petherick; Princeton University, Steven S. Gubser, Mark W. Lucianovic, Thomas J. Weisswange; University of Saskatchewan, Trevor N. Green; Simon Fraser University, Erick B. Wong; Stanford University, Svetlozar E. Nestorov; Swarthmore College, Mark D. Kernighan; Texas Tech University, Mikhail V. Shubov; Vassar College, Andrew F. Rizzo; Washington University, St. Louis, Benjamin B. Gum, Edward D. Hanson, Philip X. Wu; University of Waterloo, Daniel R. L. Brown, Eli Lapell; University of Wisconsin, Madison, Brent E. Halsey; and University of Wisconsin, Parkside, Sergey M. Ioffe.

There were 2356 individual contestants from 402 colleges and universities in Canada and the United States in the competiton of December 4, 1993. Teams were entered by 291 institutions.

The Questions Committee for the fifty-fourth competition consisted of George T. Gilbert, Texas Christian University (Chair); Fan Chung, Bellcore; and Eugene Luks, University of Oregon; they composed the problems listed below and were most prominent among those suggesting solutions.

PROBLEMS

Problem A-1. The horizontal line $y = c$ intersects the curve $y = 2x - 3x^3$ in the first quadrant as in the figure. Find c so that the areas of the two shaded regions are equal.



Problem A-2. Let $(x_n)_{n \geq 0}$ be a sequence of nonzero real numbers such that

$$x_n^2 - x_{n-1}x_{n+1} = 1, \quad \text{for } n = 1, 2, 3, \ldots .$$

Prove there exists a real number a such that $x_{n+1} = ax_n - x_{n-1}$ for all $n \geq 1$.

Problem A-3. Let $\mathscr{P}_n$ be the set of subsets of $\{1, 2, \ldots, n\}$. Let $c(n, m)$ be the number of functions $f: \mathscr{P}_n \to \{1, 2, \ldots, m\}$ such that $f(A \cap B) = \min\{f(A), f(B)\}$. Prove that

$$c(n, m) = \sum_{j=1}^{m} j^n.$$

Problem A-4. Let $x_1, x_2, \ldots, x_{19}$ be positive integers each of which is less than or equal to 93. Let $y_1, y_2, \ldots, y_{93}$ be positive integers each of which is less than or equal to 19. Prove that there exists a (nonempty) sum of some x_i's equal to a sum of some y_j's.

Problem A-5. Show that

$$\int_{-100}^{-10} \left(\frac{x^2 - x}{x^3 - 3x + 1} \right)^2 dx + \int_{\frac{1}{101}}^{\frac{1}{11}} \left(\frac{x^2 - x}{x^3 - 3x + 1} \right)^2 dx + \int_{\frac{101}{100}}^{\frac{11}{10}} \left(\frac{x^2 - x}{x^3 - 3x + 1} \right)^2 dx$$

is a rational number.

Problem A-6. The infinite sequence of 2's and 3's

$$2, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 3, 3, 2, \ldots$$

has the property that, if one forms a second sequence that records the number of 3's between successive 2's, the result is identical to the given sequence. Show that there exists a real number r such that, for any n, the nth term of the sequence is 2 if and only if $n = 1 + \lfloor rm \rfloor$ for some nonnegative integer m. (Note: $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.)

Problem B-1. Find the smallest positive integer n such that for every integer m, with $0 < m < 1993$, there exists an integer k for which

$$\frac{m}{1993} < \frac{k}{n} < \frac{m+1}{1994}.$$

Problem B-2. Consider the following game played with a deck of $2n$ cards numbered from 1 to $2n$. The deck is randomly shuffled and n cards are dealt to each of two players, A and B. Beginning with A, the players take turns discarding one of their remaining cards and announcing its number. The game ends as soon as the sum of the numbers on the discarded cards is divisible by $2n + 1$. The last person to discard wins the game. Assuming optimal strategy by both A and B, what is the probability that A wins?

Problem B-3. Two real numbers x and y are chosen at random in the interval (0,1) with respect to the uniform distribution. What is the probability that the closest integer to x/y is even? Express the answer in the form $r + s\pi$, where r and s are rational numbers.

Problem B-4. The function $K(x,y)$ is positive and continuous for $0 \le x \le 1, 0 \le y \le 1$, and the functions $f(x)$ and $g(x)$ are positive and continuous for $0 \le x \le 1$. Suppose that for all x, $0 \le x \le 1$,

$$\int_0^1 f(y)K(x,y)\,dy = g(x) \quad \text{and} \quad \int_0^1 g(y)K(x,y)\,dy = f(x).$$

Show that $f(x) = g(x)$ for $0 \le x \le 1$.

Problem B-5. Show there do not exist four points in the Euclidean plane such that the pairwise distances between the points are all odd integers.

Problem B-6. Let S be a set of three, not necessarily distinct, positive integers. Show that one can transform S into a set containing 0 by a finite number of applications of the following rule: Select two of the three integers, say x and y, where $x \le y$, and replace them with $2x$ and $y - x$.

SOLUTIONS

In the 12-tuples $(n_{10}, n_9, n_8, n_7, n_6, n_5, n_4, n_3, n_2, n_1, n_0, n_{-1})$ following each problem number below, n_i for $10 \ge i \ge 0$ is the number of students among the top 207 contestants achieving i points for the problem and n_{-1} is the number of those not submitting solutions.

A-1 (185, 2, 0, 0, 0, 0, 0, 0, 1, 0, 14, 5)

Solution. The value of c is $4/9$.

Let (b, c) denote the second intersection point. We wish to find c so that

$$\int_0^b (c - (2x - 3x^3))dx = 0.$$

This leads to $cb - b^2 + (3/4)b^4 = 0$. After substituting $c = 2b - 3b^3$ and solving, we find that $b = 2/3$ and the result follows.

A-2 (146, 21, 6, 0, 0, 0, 0, 0, 8, 1, 17, 8)

Solution 1. It is equivalent to show that

$$\frac{x_{n+1} + x_{n-1}}{x_n}$$

is independent of n. This follows (by induction) from

$$\begin{aligned}\frac{x_{n+2} + x_n}{x_{n+1}} - \frac{x_{n+1} + x_{n-1}}{x_n} &= \frac{\left(x_n x_{n+2} + x_n^2\right) - \left(x_{n+1}^2 + x_n x_{n+1}\right)}{x_n x_{n+1}} \\ &= \frac{-\left(x_{n+1}^2 - x_n x_{n+2}\right) + \left(x_n^2 - x_{n-1} x_{n+1}\right)}{x_n x_{n+1}} \\ &= \frac{-1 + 1}{x_n x_{n+1}} = 0.\end{aligned}$$

Solution 2. For all n,

$$\begin{aligned}\det\begin{pmatrix} x_{n-1} + x_{n+1} & x_n + x_{n+2} \\ x_n & x_{n+1}\end{pmatrix} &= \det\begin{pmatrix} x_{n-1} & x_n \\ x_n & x_{n+1}\end{pmatrix} + \det\begin{pmatrix} x_{n+1} & x_{n+2} \\ x_n & x_{n+1}\end{pmatrix} \\ &= -1 + 1 = 0.\end{aligned}$$

Thus, $(x_{n-1} + x_{n+1}, x_n + x_{n+2}) = c_n(x_n, x_{n+1})$ for some scalar c_n. Substituting $n - 1$ for n and then comparing the coordinate expressions, we see that $c_n = c_{n-1}$ (using $x_n \neq 0$).

Comment. In a similar manner, one can prove that if $(x_n)_{n \geq 0}$ is a sequence of nonzero real numbers such that

$$\det\begin{pmatrix} x_n & x_{n+1} & \cdots & x_{n+k} \\ x_{n+1} & x_{n+2} & \cdots & x_{n+k+1} \\ \vdots & \vdots & \vdots & \vdots \\ x_{n+k} & x_{n+k+1} & \cdots & x_{n+2k}\end{pmatrix} = cr^n \quad \text{for } n = 1, 2, 3, \ldots,$$

then there exist real numbers $a_1, \ldots, a_k$ such that

$$x_{n+k+1} = a_1 x_{n+k} + a_2 x_{n+k-1} + \cdots + a_k x_{n+1} + (-1)^k r x_n.$$

A-3 (2, 11, 24, 0, 0, 0, 0, 0, 27, 8, 54, 81)

Solution. Let $S = \{1, 2, \ldots, n\}$, and suppose that f is such a function and that $f(S) = j$. Then $f(A \cap B) = \min\{f(A), f(B)\}$ implies that the values of $f(S - \{i\})$ determine $f(A)$ for all $A \subset S$, since $f(A) = \min_{i \notin A}\{f(S - \{i\})\}$ for $A \subset S$. Conversely, arbitrary choices of $f(S - \{i\}) \leq j$ leads to such a function. Because there are j independent choices for each $f(S - \{i\})$, there are j^n functions with $f(S) = j$. Summing over the possible values of $f(S)$ yields $c(n, m) = \sum_{j=1}^{m} j^n$.

A-4 (3, 2, 0, 0, 0, 0, 0, 0, 0, 0, 44, 158)

Solution. For reasons of symmetry, let us replace 19 and 93 by m and n respectively, in the problem statement. Without loss of generality, $\sum_{i=1}^{m} x_i \geq \sum_{j=1}^{n} y_j$.

Then, for $0 \le k \le n$, there exists $f(k)$, $0 \le f(k) \le m$, such that

$$\sum_{i=1}^{f(k)} x_i \le \sum_{j=1}^{k} y_j < \sum_{i=1}^{f(k)+1} x_i.$$

Let $g(k) = \sum_{j=1}^{k} y_j - \sum_{i=1}^{f(k)} x_i$. Then, for $0 \le k \le n$, $0 \le g(k) < x_{f(k)+1} \le n$. If $g(k) = 0$ for some k, we are done. Otherwise, by the Pigeonhole Principle, there exists $k_0 < k_1$ such that $g(k_0) = g(k_1)$, in which case

$$\sum_{i=f(k_0)+1}^{f(k_1)} x_i = \sum_{j=k_0+1}^{k_1} y_j.$$

A-5 (3, 3, 0, 0, 0, 0, 0, 0, 1, 16, 37, 147)

Solution 1. Observe first that the roots of $x^3 - 3x + 1$ can be isolated away from the given intervals; that is, there are sign changes in the intervals $[-2, -1]$, $[1/3, 1/2]$, $[3/2, 2]$. Hence the integrand is defined and continuous throughout.

Set

$$f(t) = \int_{-100}^{t} \left(\frac{x^2 - x}{x^3 - 3x + 1}\right)^2 dx + \int_{\frac{1}{101}}^{1/(1-t)} \left(\frac{x^2 - x}{x^3 - 3x + 1}\right)^2 dx$$
$$+ \int_{\frac{101}{100}}^{1-1/t} \left(\frac{x^2 - x}{x^3 - 3x + 1}\right)^2 dx$$

for $-100 \le t \le -10$. We wish to compute $f(-10)$. By the Fundamental Theorem of Calculus,

$$f'(t) = Q(t) + Q\left(\frac{1}{1-t}\right)\frac{1}{(1-t)^2} + Q\left(1 - \frac{1}{t}\right)\frac{1}{t^2}$$

where $Q(x) = ((x^2 - x)/(x^3 - 3x + 1))^2$. We find that $Q(1/(1 - t)) = Q(1 - 1/t) = Q(t)$, so that

$$f(-10) = \int_{-100}^{-10} \left(\frac{x^2 - x}{x^3 - 3x + 1}\right)^2 \left(1 + \frac{1}{x^2} + \frac{1}{(1-x)^2}\right) dx.$$

But, noting that

$$\frac{1}{Q(x)} = \left(x + 1 - \frac{1}{x} - \frac{1}{x-1}\right)^2,$$

we see that the last integral is of the form $\int du/(u^2)$. Hence, its value is

$$-\frac{x^2 - x}{x^3 - 3x + 1}\Bigg|_{-100}^{10},$$

which is rational.

Solution 2. By the substitutions $x = -1/(t - 1)$ and $x = 1 - 1/t$, the integrals over $[1/101, 1/11]$ and $[101/100, 11/10]$ are respectively converted into integrals over $[-100, -10]$. In the course of this it is seen that the function

$$\left(\frac{x^2 - x}{x^3 - 3x + 1}\right)^2$$

is invariant under each of the substitutions $x \to 1 - 1/x$ and $x \to -1/(x-1)$ (which, in fact, are inverses of one another). Hence the sum of the three given integrals is expressible as

$$\int_{-100}^{-10}\left(\frac{x^2 - x}{x^3 - 3x + 1}\right)^2\left(1 + \frac{1}{x^2} + \frac{1}{(x-1)^2}\right)dx.$$

The solution continues as in the first solution.

Comment. It can be shown, more generally, that if $f(x)$ is a rational polynomial of degree at most 4, then

$$\int_{-100}^{-10}\frac{f(x)}{(x^3 - 3x + 1)^2}dx + \int_{1/101}^{1/11}\frac{f(x)}{(x^3 - 3x + 1)^2}dx + \int_{101/100}^{11/10}\frac{f(x)}{(x^3 - 3x + 1)^2}dx$$

is rational.

A-6 (0, 1, 0, 0, 0, 0, 1, 3, 3, 11, 36, 152)

Solution. We show that the conclusion holds with $r = 2 + \sqrt{3}$.

Assuming the result, we first derive the value of r. Observe that, asymptotically, the proportion of 2's in the first n terms is $1/r$. Thus, assuming there are about m 2's in the first $n \approx rm$ terms, there should be about $(r-1)m$ 3's. These numbers give the approximate number of 3's in the intervals following the first n 2's, namely $2m + 3(r-1)m = (3r-1)m$. Hence, the proportion of 2's in the first $rm + (3r-1)m = (4r-1)m$ terms is $rm/(4r-1)m = r/(4r-1)$. So we want r to satisfy $1/r = r/(4r-1)$, or $r^2 - 4r + 1 = 0$. Since r must exceed 1, $r = 2 + \sqrt{3}$.

Note that substituting $m = 0$ into the formula yields the first 2 and $m = 1$ the next. Assume the jth 2 is in position $1 + \lfloor r(j-1)\rfloor$ for $j \le m$. Solving $1 + \lfloor r(j-1)\rfloor \le m < 1 + \lfloor rj\rfloor$ yields $j = \lfloor m/(2+\sqrt{3})\rfloor + 1$ 2's among the first m numbers of the sequence. Thus the $(m+1)$st 2 is in position

$$\begin{aligned}(m+1) &+ 2(\lfloor m/(2+\sqrt{3})\rfloor + 1) + 3(m - \lfloor m/(2+\sqrt{3})\rfloor - 1)\\ &= 4m - \lfloor m/(2+\sqrt{3})\rfloor\\ &= \lfloor 4m - (2-\sqrt{3})m\rfloor = \lfloor (2+\sqrt{3})m\rfloor.\end{aligned}$$

The claim follows by induction.

B-1 (83, 29, 9, 0, 0, 0, 0, 0, 6, 10, 38, 32)

Solution. First, it is easily verified that

$$\frac{m}{1993} < \frac{2m+1}{1993+1994} < \frac{m+1}{1994},$$

so $n = 1993 + 1994 = 3987$ suffices. Now consider $m = 1992$ and suppose

$$\frac{1992}{1993} < \frac{k}{n} < \frac{1993}{1994}.$$

Since $x/(x+1)$ is strictly increasing for $x > 0$, we must have $k \le n - 2$ (note: $n > 1994$). However,

$$\frac{1992}{1993} < \frac{n-2}{n}$$

implies $3986 < n$, so $n \ge 3987$, completing the proof.

B-2 (89, 0, 1, 0, 0, 0, 0, 0, 7, 3, 42, 65)

Solution. The probability that A wins is 0.

Clearly, A cannot win on the first turn. Assume B is to play, and that the total of announced numbers is T, and that A has cards $x_1, x_2, \dots, x_k$, and B has cards $y_1, y_2, \dots, y_{k+1}$. Because the integers $T + y_1, \dots, T + y_{k+1}$ have distinct remainders upon division by $2n+1$, at least one has a remainder other than $2n + 1 - x_1, \dots, 2n + 1 - x_k$. If B discards that y_i, it is impossible for A's next discard to make the total divisible by $2n+1$. Therefore, A cannot win under optimal play by B.

B-3 (111, 16, 13, 0, 0, 0, 0, 0, 4, 20, 10, 33)

Solution. The limit is $(5-\pi)/4$ (that is, when $r = 5/4$, $s = -1/4$).

Note that the probability that x/y is exactly half an odd integer is 0, so we may safely ignore this possibility.

For any choice of x, the closest integer to x/y is even if either $x/y < .5$ or $2n - .5 < x/y < 2n + .5$ for some positive integer n.

The event $x/y < .5$, or $2x < y$, can occur only if $x < .5$ Thus its probability is

$$\int_0^{.5} (1 - 2x)\, dx = \tfrac{1}{4}.$$

For a positive integer n, the probability that $2n - .5 < x/y < 2n + .5$, i.e., that $2x/(4n+1) < y < 2x/(4n-1)$, is

$$\int_0^1 \left(\frac{2x}{4n-1} - \frac{2x}{4n+1} \right) dx = \frac{1}{4n-1} - \frac{1}{4n+1}.$$

Summing from $n = 1$ to ∞, we get

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{2k+1} = 1 - \arctan 1 = 1 - \frac{\pi}{4}.$$

The total probability then is $1/4 + 1 - \pi/4 = (5-\pi)/4$.

B-4 (0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 61, 144)

Solution. For $0 \le x \le 1$,

$$\begin{aligned} f(x) &= \int_0^1 g(t) K(x,t)\, dt \\ &= \int_0^1 \int_0^1 f(y) K(t,y) K(x,t)\, dy\, dt \\ &= \int_0^1 f(y) L(x,y)\, dy \end{aligned}$$

where

$$L(x, y) = \int_0^1 K(x, t) K(t, y)\, dt$$

for $0 \le x \le 1, 0 \le y \le 1$.

Similarly,

$$g(x) = \int_0^1 g(y) L(x, y) dy.$$

Since

$$\int_0^1 \frac{L(x, y) f(y)}{f(x)} dy = 1$$

and

$$\int_0^1 \left(\frac{L(x, y) f(y)}{f(x)} \right) \frac{g(y)}{f(y)} dy = \frac{g(x)}{f(x)},$$

for $0 \le x \le 1$, it follows that $g/f = c$, a constant. Then

$$\begin{aligned} g(x) = cf(x) &= c\int_0^1 g(y) K(x, y) dy \\ &= c^2 \int_0^1 f(y) K(x, y) dy \\ &= c^2 g(x). \end{aligned}$$

Therefore, $c = 1$.

B-5 (6, 1, 0, 0, 0, 0, 0, 0, 1, 7, 65, 127)

Solution. Suppose there were 4 such points. Locate one point at the origin and let $\vec{v}_1, \vec{v}_2, \vec{v}_3$ be vectors from the origin to the other three. Since $\vec{v}_i \cdot \vec{v}_i$, and $|\vec{v}_i - \vec{v}_j|^2 = \vec{v}_i \cdot \vec{v}_i - 2\vec{v}_i \cdot \vec{v}_j + \vec{v}_j \cdot \vec{v}_j$, for $i \neq j$, are squares of odd integers we know $\vec{v}_i \cdot \vec{v}_i$ as well as $2\vec{v}_i \cdot \vec{v}_j$, for $j \neq i$ are integers congruent to 1 (mod 8).

Clearly no three points can be collinear. Hence $\vec{v}_3 = x\vec{v}_1 + y\vec{v}_2$ for some scalars x and y. Then

$$\begin{aligned} 2\vec{v}_1 \cdot \vec{v}_3 &= 2x\vec{v}_1 \cdot \vec{v}_1 + 2y\vec{v}_1 \cdot \vec{v}_2 \\ 2\vec{v}_2 \cdot \vec{v}_3 &= 2x\vec{v}_2 \cdot \vec{v}_1 + 2y\vec{v}_2 \cdot \vec{v}_2 \\ 2\vec{v}_3 \cdot \vec{v}_3 &= 2x\vec{v}_3 \cdot \vec{v}_1 + 2y\vec{v}_3 \cdot \vec{v}_2 \end{aligned} \tag{1}$$

Since $\vec{v}_1$ is not a scalar multiple of $\vec{v}_2$,

$$\det \begin{pmatrix} \vec{v}_1 \cdot \vec{v}_1 & \vec{v}_1 \cdot \vec{v}_2 \\ \vec{v}_2 \cdot \vec{v}_1 & \vec{v}_2 \cdot \vec{v}_2 \end{pmatrix} > 0$$

so that the first two equations in (1) have a unique rational solution for x, y, say $x = X/D$, $y = Y/D$, where X, Y, and D are integers. We may assume $\gcd(X, Y, D) = 1$. Then multiplying (1) through by D we have

$$\begin{aligned} D &\equiv 2X + Y \pmod 8 \\ D &\equiv X + 2Y \pmod 8 \\ 2D &\equiv X + Y \pmod 8 \end{aligned}$$

Adding the first two congruences and subtracting the third gives $2X + 2Y \equiv 0 \pmod{8}$, so that, by the third congruence D is even. But then the first two congruences force Y and X, respectively, to be even, a contradiction.

B-6 (2, 0, 0, 0, 0, 0, 0, 0, 1, 2, 59, 143)

Solution. Say the numbers are a, b, c. First, we reduce to the case that exactly one of a, b, c is odd. Namely: (i) if two are odd, apply the rule with those two, and none is odd; (ii) if none is odd, divide all numbers by 2 and apply induction; (iii) if three are odd, apply the rule once, and exactly one is odd. Once exactly one is odd, this will remain so.

Say a is odd and b and c even. We aim to make the power of 2 dividing $b + c$ as large as possible. If b and c have the same number of factors of 2, then applying the rule to those two will yield both divisible by a higher power of 2, or one will have fewer factors of 2 than the other. Since $b + c$ is constant here, after a finite number of applications of the rule, b and c will not have the same number of factors of 2. Also, it is easy to see that, possibly after some additional moves, one has either $bc = 0$ (in which case one stops), or, the one of b and c divisible by the smaller power of 2 is also smaller; say it is b, so that $b < c$.

Case 1: $a > b$. Now work with a, b. Then a remains odd, b is doubled, and $b + c$ is divisible by a higher power of 2.

Case 2: $a < b$. Apply the rule first to a and b, and then to $b - a$ and c. (Note that $c > b > b - a$.) One obtains

$$\begin{array}{ccc} a & b & c \\ 2a & b - a & c \\ 2a & 2b - 2a & c - b + a \end{array}$$

Now the odd number is $c - b + a$, and the sum of the even numbers is $2b$, which has more factors of 2 than $b + c$.

Klosinski:
Department of Mathematics
Santa Clara University
Santa Clara, CA 95053

Alexanderson:
Department of Mathematics
Santa Clara University
Santa Clara, CA 95053

Larson:
Department of Mathematics
St. Olaf College
Northfield, MN 55057