The Fifty-Fifth William Lowell Putnam Mathematical Competition

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The following results of the fifty-fifth William Lowell Putnam Mathematical Competition, held on December 3, 1994, have been determined in accordance with the governing regulations. This annual contest is supported by the William Lowell Putnam Prize Fund for the Promotion of Scholarship, left by Mrs. Putnam in memory of her husband, and is held under the auspices of the Mathematical Association of America.

The first prize, $7,500, was awarded to the Department of Mathematics at Harvard University. The members of the winning team were: Kiran S. Kedlaya, Lenhard L. Ng, and Dylan P. Thurston; each was awarded a prize of $500.

The second prize, $5,000, was awarded to the Department of Mathematics at Cornell University. The members of the winning team were Jeremy L. Bem, Robert D. Kleinberg, and Mark Krosky; each was awarded a prize of $400.

The third prize, $3,000, was awarded to the Department of Mathematics at the Massachusetts Institute of Technology. The members of the winning team were Henry L. Cohn, Adam W. Meyerson, and Thomas A. Weston; each was awarded a prize of $300.

The fourth prize, $2,000, was awarded to the Department of Mathematics at Princeton University. The members of the winning team were William R. Mann, Joël E. Rosenberg, and Michail Sunitsky; each was awarded a prize of $200.

The fifth prize, $1,000, was awarded to the Department of Mathematics of the University of Waterloo. The members of the winning team were Ian A. Goldberg, Peter L. Milley, and Kevin Purbhoo; each was awarded a prize of $100.

The five highest ranking individual contestants, in alphabetical order, were Jeremy L. Bem, Cornell University; J. P. Grossman, University of Toronto; Kiran S. Kedlaya, Harvard University; William R. Mann, Princeton University; and Lenhard L. Ng, Harvard University. Each of these was designated a Putnam Fellow by the Mathematical Association of America and awarded a prize of $1,000, by the Putnam Prize Fund.

The next five highest ranking contestants, in alphabetical order, were Soundararajan Kannan, University of Michigan, Ann Arbor; David L. Savitt, University of British Columbia; Daniel K. Schepler, Washington University, St. Louis; Noam M. Shazeer, Duke University; and Hong Zhou, Harvard Univer-sity; each was awarded a prize of $500.

The next six highest ranking contestants, in alphabetical order, were Alexandru D. Ionescu, Massachusetts Institute of Technology; Robert D. Kleinberg, Cornell University; Jacob A. Rasmussen, Princeton University; Andrew H. Schultz, Johns Hopkins University; Dylan P. Thurston, Harvard University; and Zhaohui Zhang, Yale University; each was awarded a prize of $250.

The next nine highest ranking contestants, in alphabetical order, were Henry L. Cohn, Massachusetts Institute of Technology; Ian A. Goldberg, University of Waterloo; Adam Kalai, Harvard University; Serban M. Nacu, Harvard University; Joel E. Rosenberg, Princeton University; Mikhail V. Shubov, Texas Tech University; Jade P. Vinson, Washington University, St. Louis; Stephen S. Wang, Harvard University; and Jonathan L. Weinstein, Harvard University. Each was awarded a prize of $100.

The following teams, named in alphabetical order, received honorable mention: University of Nebraska, Lincoln, with team members Scott Annin, Igor V. Pavlovsky, and Eric M. Smith; New York University, with team members Igor Berger, Yevgeniy Dodis, and Mikhail Kogan; University of Toronto, with team members J. P. Grossman, Edward Leung, and Naoki Sato; Washington University, St. Louis, with team members Ben Gum, Daniel K. Schepler, and Jade P. Vinson; and Yale University, with team members Gautam Chinta, Matthew Frank, and Zhaohui Zhang.

Honorable mention was achieved by the following thirty individuals named in alphabetical order: Jared E. Anderson, University of Victoria; Federico Ardila, Massachusetts Institute of Technology; Bradley S. Bart, University of Waterloo; Ruth A. Britto-Pacumio, Massachusetts Institute of Technology; Robert H. Cheng, University of British Columbia; Yevgeniy Dodis, New York University; Ron D. Dror, Rice University; Alex Heneveld, Princeton University; Randy W. Ho, University of Arizona; Jason A. Howald, Miami University; Sergey M. Ioffe, Massachusetts Institute of Technology; Dean W. Jens, University of Chicago; Joanna L. Karczmarek, Queen's University; Mikhail Kogan, New York University; Botond Kőszegi, Harvard University; Mark Krosky, Cornell University; Daniel T. Martin, Carleton College; Olexei Ivanovich Motrunich, University of Missouri, Columbia; Akira Negi, University of North Carolina, Chapel Hill; An T. Nguyen, University of Texas, Austin; Royce Y. Peng, Harvard University; Kevin Purbhoo, University of Waterloo; Lawrence P. Roberts, Washington University, St. Louis; NNaoki Sato, University of Toronto; Sam Spencer, Rice University; Jason M. Starr, University of California, Berkeley; Mark A. Van Raamsdonk, University of British Columbia; David R. Wasserman, University of California, San Diego; Thomas A. Weston, Massachusetts Institute of Technology; and Jeffrey S. Willson, University of Chicago.

The other individuals who achieved ranks among the top 107, in alphabetical order of their schools, were: Brown University, Andrew Brecher; California Institute of Technology, Wei-Hwa Huang, Roman Muchnik; California Polytechnic State University, San Luis Obispo, Robert B. Mathews; University of California, Santa Barbara, Aaron S. Cohen; Carleton College, Curtis Z. Mitchell; Case Western Reserve University, Neil A. Rubin; Colgate University, Jean-François R. Lafont; Dartmouth College, Yuan Shen; Duke University, Robert R. Schneck; Harvard University, Manjul Bhargava, Dean R. Chung, Joe B. Fendel, Sergey V. Levin, Paul Li, Harrison K. Tsai, Jiří J. L. Vaníček; Harvey Mudd College, Aaron F. Archer, Kan Yasuda; University of Illinois, Champaign-Urbana, Ivan Auramovic, Kwong Shing Lin; Massachusetts Institute of Technology, Adam W. Meyerson, Michael B. Schulz, Michael R. Tehranchi, Aleksey Zinger; McGill University, Jacob Eliosoff; University of Nebraska, Lincoln, Eric M. Smith; New York University, Igor Berger; University of North Carolina, Chapel Hill, Paul E. Rube; Northwestern University, Carol R. James; Princeton University, Paul J. Ellis, Michael J. Goldberg, Mark W. Lucianovic; Queen's University, Peter Gregory Zion; Reed College, Gerald D. Larson; Rice University, Ashley M.

Reiter; University of Saskatchewan, Trevor N. Green; University of the South, Qingshan Luo; Stanford University, Robert G. Au, Heyning A. Cheng, Loren L. Looger; Suffolk University, Anna V. Petrovskaya; Vanderbilt University, Jason D. Hughes; Washington University, St. Louis, Ian F. Pulizzotto, Erik N. Vee; University of Waterloo, Jason P. Bell, Jie J. Lou, Peter L. Milley, Lousindi R. Sabourin; Williams College, Jason R. Schweinsberg, Edward W. Welsh; and Yale University, Matthew Frank.

The Elizabeth Lowell Putnam Prize, named for the wife of William Lowell Putnam and to be "awarded periodically to a woman whose performance on the Competition has been deemed particularly meritorious," is awarded this year to Ruth A. Britto-Pacumio of the Massachusetts Intitute of Technology. The winner is awarded a prize of $500.

There were 2,314 individual contestants from the 410 colleges and universities in Canada and the United States in the competition of December 3, 1994. Teams were entered by 284 institutions. The Questions Committee for the fifty-fifth competition consisted of Eugene M. Luks, University of Oregon, chair; Fan Chung, Bellcore; and Mark I. Krusemeyer, Carleton College; they composed the problems listed below and were most prominent among those suggesting solutions.

PROBLEMS

Problem A-1. Suppose that a sequence $a_1, a_2, a_3, \ldots$ satisfies $0 < a_n \leq a_{2n} + a_{2n+1}$ for all $n \geq 1$. Prove that the series $\Sigma_{n=1}^{\infty} a_n$ diverges.

Problem A-2. Let A be the area of the region in the first quadrant bounded by the line $y = \frac{1}{2}x$, the x-axis, and the ellipse $\frac{1}{9}x^2 + y^2 = 1$. Find the positive number m such that A is equal to the area of the region in the first quadrant bounded by the line $y = mx$, the y-axis, and the ellipse $\frac{1}{9}x^2 + y^2 = 1$.

Problem A-3. Show that if the points of an isosceles right triangle of side length 1 are each colored with one of four colors, then there must be two points of the same color which are at least a distance $2 - \sqrt{2}$ apart.

Problem A-4. Let A and B be 2×2 matrices with integer entries such that A, $A + B$, $A + 2B$, $A + 3B$, and $A + 4B$ are all invertible matrices whose inverses have integer entries. Show that $A + 5B$ is invertible and that its inverse has integer entries.

Problem A-5. Let $(r_n)_{n \geq 0}$ be a sequence of positive real numbers such that $\lim_{n \to \infty} r_n = 0$. Let S be the set of numbers representable as a sum

$$r_{i_1} + r_{i_2} + \cdots + r_{i_{1994}},$$

with $i_1 < i_2 < \cdots < i_{1994}$. Show that every nonempty interval (a, b) contains a nonempty subinterval (c, d) that does not intersect S.

Problem A-6. Let $f_1, f_2, \ldots, f_{10}$ be bijections of the set of integers such that for each integer n, there is some composition $f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_m}$ of these functions (allowing repetitions) which maps 0 to n. Consider the set of 1024 functions

$$\mathscr{F} = \{f_1^{e_1} \circ f_2^{e_2} \circ \cdots \circ f_{10}^{e_{10}}\},$$

$e_i = 0$ or 1 for $1 \leq i \leq 10$. (f_i^0 is the identity function and $f_i^1 = f_i$.) Show that if A

is any nonempty finite set of integers, then at most 512 of the functions in $\mathcal{F}$ map A to itself.

Problem B-1. Find all positive integers that are within 250 of exactly 15 perfect squares.

Problem B-2. For which real numbers c is there a straight line that intersects the curve

$$y = x^4 + 9x^3 + cx^2 + 9x + 4$$

in four distinct points?

Problem B-3. Find the set of all real numbers k with the following property: For any positive, differentiable function f that satisfies $f'(x) > f(x)$ for all x, there is some number N such that $f(x) > e^{kx}$ for all $x > N$.

Problem B-4. For $n \geq 1$, let d_n be the greatest common divisor of the entries of $A^n - I$, where

$$A = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} \quad \text{and} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Show that $\lim_{n \to \infty} d_n = \infty$.

Problem B-5. For any real number α, define the function $f_\alpha(x) = \lfloor \alpha x \rfloor$. Let n be a positive integer. Show that there exists an α such that for $1 \leq k \leq n$,

$$f_\alpha^k(n^2) = n^2 - k = f_{\alpha^k}(n^2).$$

Problem B-6. For any integer a, set

$$n_a = 101a - 100 \cdot 2^a.$$

Show that for $0 \leq a, b, c, d \leq 99$, $n_a + n_b \equiv n_c + n_d \pmod{10100}$ implies $\{a, b\} = \{c, d\}$.

SOLUTIONS. In the 12-tuples $(n_{10}, n_9, n_8, n_7, n_6, n_5, n_4, n_3, n_2, n_1, n_0, n_{-1})$ following each problem number below, n_i for $10 \geq i \geq 0$ is the number of students among the top 206 contestants achieving i points for the problem and n_{-1} is the number of those not submitting solutions.

A-1 (59, 59, 54, 21, 0, 0, 0, 0, 8, 0, 3, 2)

Solution. Let $b_1 = a_1$, $b_2 = a_2 + a_3$, $b_3 = a_4 + a_5 + a_6 + a_7$, and in general, $b_n = a_{2^{n-1}} + a_{2^{n-1}+1} + \cdots + a_{2^n - 1}$. AN easy induction, using the condition $a_n \leq a_{2n} + a_{2n+1}$ shows that $b_n \leq b_{n+1}$ for all $n \geq 1$. Thus, for any positive integer t,

$$\sum_{n=1}^{\infty} a_n > \sum_{n=1}^{2^t - 1} a_n = \sum_{n=1}^{t} b_n \geq tb_1 = ta_1.$$

This shows that $\sum_{n=1}^{\infty} a_n$ diverges.

A-2 (169, 3, 2, 0, 0, 0, 0, 0, 1, 3, 22, 6)

Solution. The linear transformation given by $x_1 = \frac{1}{3}x$, $y_1 = y$ transforms the region R bounded by $y = \frac{1}{2}x$, the x-axis, and the ellipse $\frac{1}{9}x^2 + y^2 = 1$ into the region R' bounded by $y_1 = \frac{3}{2}x_1$, the x_1-axis, and the circle $x_1^2 + y_1^2 = 1$; it also transforms the region S bounded by $y = mx$, the y-axis, and $\frac{1}{9}x^2 + y^2 = 1$ into the region S' bounded by $y_1 = 3mx_1$, the y_1-axis, and the circle. Since all areas are multiplied by the same (nonzero) factor under the transformation, R and S have the same area if and only if R' and S' have the same area. However, we can see by symmetry about the line $y_1 = x_1$ that this happens if and only if $3m = \frac{2}{3}$, that is, $m = \frac{2}{9}$.

A-3 (0, 10, 67, 0, 0, 0, 0, 0, 30, 31, 40, 28)

Solution. Suppose the vertices of the isosceles right triangle are $(0,0)$, $(1,0)$, $(0,1)$. Suppose the points of the triangle can be colored in four colors such that points of the same color are always less that a distance $2 - \sqrt{2}$ apart. Then the four points $(0,1)$, $(0, \sqrt{2} - 1)$, $(\sqrt{2} - 1, 0)$, $(1,0)$ must have different colors, say colors A, B, C, D respectively. The point $(0,0)$ must be of color B or C. Without loss of generality, say $(0,0)$ is of color B. Then the point $(\sqrt{2} - 1, 2 - \sqrt{2})$ is of distance at least $2 - \sqrt{2}$ to points of each of the four colors, and this is impossible.

A-4 (12, 17, 20, 0, 0, 0, 0, 0, 15, 3, 43, 96)

Solution. A matrix C with integer entries has an inverse with integer entries if and only if $\det C = \pm 1$. Therefore, if we consider the function f defined by $f(x) = \det(A + xB)$, we know that the five values $f(0)$, $f(1)$, $f(2)$, $f(3)$, and $f(4)$ must all be 1 or -1, so f takes on at least one of those values three or more times. However, $f(x)$ is a polynomial of degree ≤ 2 in x, and so f can only take on a value more than twice if f is constant. Thus $f(x)$ is one of the constants 1 and -1; in particular, $\det(A + 5B) = \pm 1$, so $A + 5B$ has an inverse with integer entries.

A-5 (20, 13, 4, 0, 0, 0, 0, 0, 6, 2, 57, 104)

Solution 1. It suffices to show that any sequence in S contains a monotonically nonincreasing subsequence. For then, letting $(t_n)_{n \geq 0}$ be any strictly inncreasing sequence within (a, b), some (in fact, all but a finite number) of the intersections $S \cap (t_n, t_{n+1})$ would have to be empty, otherwise one could form a strictly increasing sequence $(s_n)_{n \geq 0}$ by taking $s_n \in S \cap (t_n, t_{n+1})$.

Let $(s_n)_{n \geq 0}$ be a sequence in S. For $n = 0, 1, 2, \ldots$ write

$$s_n = r_{f(n,1)} + r_{f(n,2)} + \cdots + r_{f(n,1994)} \qquad \text{with} \quad f(n,1) < f(n,2) < \cdots < f(n,1994).$$

The sequence $(r_{f(n,1)})_{n \geq 0}$ has a monotonically nonincreasing subsequence (since $(r_n)_{n \geq 0}$ is a positive sequence converging to 0). Thus we may replace $(s_n)_{n \geq 0}$ by a subsequence for which $(r_{f(n,1)})_{n \geq 0}$ is monotonically nonincreasing. In a similar fashion, we pass to subsequences so that, successively, each of $(r_{f(n,2)})_{n \geq 0}$, $(r_{f(n,3)})_{n \geq 0}, \ldots, (r_{f(n,1994)})_{n \geq 0}$ may be assumed to be monotonically nonincreasing. The resulting $(s_n)_{n \geq 0}$ is monotonically nonincreasing.

Solution 2. Let C be the set $\{r_n\}_{n \geq 0} \cup \{0\}$. Since C is compact, the set S' of numbers representable as a sum of 1994 elements of C is also compact (for example, it is a continuous image of C^{1994}). Clearly $S \subseteq S'$.

Let (a, b) be a nonempty open interval. Since S' is countable, $(a, b) \setminus S'$ is nonempty; it is open since S' is closed. Hence $(a, b) \setminus S'$ includes a nonempty open interval.

Comment: This proof generalizes to give the same conclusion for any convergent sequence $(r_n)_{n \geq 0}$.

A-6 (5, 8, 10, 0, 0, 0, 0, 0, 7, 4, 34, 138)

Solution. Let A be a nonempty finite subset of the integers $\mathbf{Z}$. By the Pigeonhole Principle, any bijection of $\mathbf{Z}$ which maps A to itself must be a bijection when restricted to A; in particular, its inverse also maps A to itself. Note tha tnot all the bijections $f_1, f_2, \ldots, f_{10}$ can map A to itself, for otherwise if $0 \in A$ we could not map 0 to any $n \notin A$ by a composition $f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_m}$, while if $0 \notin A$, we could not map 0 to any $n \in A$ by such a composition.

Let k be the smallest integer such that f_k does *not* map A to itself, and suppose that more than 512 of the functions $\mathcal{F}$ map A to itself. We can write $\mathcal{F}$ as a disjoint union of unordered pairs of functions such that two compositions $f_1^{e_1} \circ f_2^{e_2} \circ \cdots \circ f_{10}^{e_{10}}$ and $f_1^{d_1} \circ f_2^{d_2} \circ \cdots \circ f_{10}^{d_{10}}$ are in the same pair when they differ only in the k-th exponent; that is, when $e_i = d_i$ for $i \neq k$. By the Pigeonhole Principle, there is then at least one of these 512 pairs in which both functions map A to itself. Since all f_l with $l > k$ also map A to itself, we can use composition with the inverses of f_l, as needed, to conclude that for some $e_1, \ldots, e_{k-1}$, $F_1 = f_1^{e_1} \circ f_2^{e_2} \circ \cdots \circ f_{k-1}^{e_{k-1}}$ and $F_2 = f_1^{e_1} \circ f_2^{e_2} \circ \cdots \circ f_{k-1}^{e_{k-1}} \circ f_k$ both map A to itself. But then $F_1^{-1} \circ F_2 = f_k$ also maps A to itself, a contradiction.

B-1 (45, 26, 57, 0, 0, 0, 0, 0, 42, 28, 6, 2)

Solution. Answer: $\{N | 315 \leq N \leq 325 \text{ or } 332 \leq N \leq 350\}$.

Assume $N > 0$ is within 250 of the 15 squares $m^2, (m+1)^2, \ldots, (m+14)^2$, where we can take $m \geq 0$. In fact, m will then be positive, otherwise N would be within 250 of the additional square 225. We have the necessary and sufficient conditions

$$(m+14)^2 \leq N + 250 \leq (m+15)^2 - 1,$$

$$(m-1)^2 + 1 \leq N - 250 \leq m^2.$$

Subtracting (reversing inequalities in the second line), we get

$$28m + 196 \leq 500 \leq 32m + 222,$$

which implies $m = 9$ or 10.

If $m = 9$,

$$23^2 \leq N + 250 \leq 24^2 - 1,$$

$$8^2 + 1 \leq N - 250 \leq 9^2,$$

or $315 \leq N \leq 325$.

If $m = 10$,

$$24^2 \leq N + 250 \leq 25^2 - 1,$$

$$9^2 + 1 \leq N - 250 \leq 10^2,$$

or $332 \leq N \leq 350$.

B-2 (28, 8, 49, 0, 0, 0, 0, 0, 56, 10, 39, 16)

Solution. Answer: For the real numbers c with $c < 243/8$.

The constant term and the coefficient of x in a quartic $p(x)$ are irrelevant in determining whether there is a line intersection $y = p(x)$ in four points. We may also replace $p(x)$ by $p(x - \alpha)$ for any real α. Thus, we may replace the given quartic $p(x) = x^4 + 9x^3 + cx^2 + 9x + 4$ with $p(x - 9/4) = x^4 + (c - 243/8)x^2 + \cdots$, and drop the last two coefficients (we need never calculate them).

The problem then is to determine the values of c for which there is a straight line that intersects $y = x^4 + (c - 243/8)x^2$ in four distinct points. The result is now apparent from the shapes of the curves $y = x^4 + ax^2$. For example, we may note that when $a < 0$, this "W-shaped" curve has a relative maximum at $x = 0$, so that horizontal lines $y = -\varepsilon$ for small positive ε intersect the curve in four points, while for $a \geq 0$, the curve is always concave upward, so that no line can intersect it in more than two points.

B-3 (27, 10, 8, 5, 0, 0, 0, 2, 45, 49, 15, 45)

Solution. The desired set is $(-\infty, 1)$.

To show this, first note that if $k > 1$ were in the set, then $k = 1$ would also be in the set. However, if f is any function of the form $f(x) = g(x)e^x$, where g is a positive, increasing, differentiable function bounded by 0 and 1 (for example, $g(x) = (1/\pi)\arctan x + \frac{1}{2}$), we have $f'(x) = e^x(g'(x) + g(x)) > f(x)$ and $f(x) < e^x$ for all x, so $k = 1$ is not in the set.

On the other hand, if $f'(x) > f(x)$ for all x, then (since f is positive) we have

$$\frac{f'(x)}{f(x)} > 1 \quad \text{for all } x,$$

$$\int_0^x \frac{f'(t)}{f(t)}\,dt > \int_0^x 1\,dt \quad \text{for all } x \geq 0,$$

$$\log(f(x)) > x + \log(f(0)) \quad \text{for all } x \geq 0,$$

$$f(x) > f(0)e^x \quad \text{for all } x \geq 0.$$

If k is any number less than 1, then for large enough x we will have $f(0)e^x > e^{kx}$ (since $f(0)$ is positive), which shows that k is in the set.

B-4 (15, 1, 4, 0, 0, 0, 0, 0, 5, 22, 71, 88)

Solution 1. From experimentation (and then an easy induction on n) we see that A^n has the form

$$A^n = \begin{pmatrix} a_n & b_n \\ 2b_n & a_n \end{pmatrix}$$

with a_n odd, and, since $\det A^n = 1$, we have $a_n^2 - 1 = 2b_n^2$. Thus $a_n - 1$ divides $2b_n^2$, so that $d_n = \gcd(a_n - 1, b_n) \geq \sqrt{(a_n - 1)/2}$. Since $\lim_{n\to\infty} a_n = \infty$ (e.g., $a_n > 3a_{n-1}$), the result follows.

Solution 2. Define the sequence $r_0, r_1, r_2, \ldots$ by $r_0 = 0$, $r_1 = 1$, and $r_k = 6r_{k-1} - r_{k-2}$ for $k > 1$. We first show by induction on k that

$$A^n = I = r_{k+1}(A^{n-k} - A^k) - r_k(A^{n-k-1} - A^{k+1}) \quad \text{for } k \geq 0. \tag{1}$$

This is clear for $k = 0$ and, for the inductive step, using $A^2 - 6A + I = 0$ (the characteristic equation), we have

$$\begin{aligned} &r_{k+1}(A^{n-k} - A^k) - r_k(A^{n-k-1} - A^{k+1}) \\ &\quad = r_{k+1}((6A^{n-k-1} - A^{n-k-2}) - (6A^{k+1} - A^{k+2})) - r_k(A^{n-k-1} - A^{k+1}) \\ &\quad = (6r_{k+1} - r_k)(A^{n-k-1} - A^{k+1}) - r_{k+1}(A^{n-k-2} - A^{k+2}) \\ &\quad = r_{k+2}(A^{n-k-1} - A^{k+1}) - r_{k+1}(A^{n-k-2} - A^{k+2}). \end{aligned}$$

Applying (1) with $k = \lfloor n/2 \rfloor$, we obtain

$$A^n - I = \begin{cases} r_{n/2}(A^{n/2+1} - A^{n/2-1}), & \text{if } n \text{ is even,} \\ (r_{(n+1)/2} + r_{(n-1)/2})(A^{(n+1)/2} - A^{(n-1)/2}), & \text{if } n \text{ is odd.} \end{cases}$$

In either case, the entries of $A^n - I$ have a common factor that $\to \infty$ since $\lim_{n\to\infty} r_n = \infty$ (e.g., $r_n > 5r_{n-1}$ for $n > 1$).

Solution 3. We know that the entries of A^n are each of the form $\alpha_1\lambda_1^n + \alpha_2\lambda_2^n$ where $\lambda_1 = 3 + 2\sqrt{2}$ and $\lambda_2 = 3 - 2\sqrt{2}$ (the eigenvalues of A). So, using the entries for $n = 1, 2$, we derive

$$A^n = \begin{pmatrix} \dfrac{\lambda_1^n + \lambda_2^n}{2} & \dfrac{\lambda_1^n - \lambda_2^n}{2\sqrt{2}} \\ \dfrac{\lambda_1^n - \lambda_2^n}{\sqrt{2}} & \dfrac{\lambda_1^n + \lambda_2^n}{2} \end{pmatrix}.$$

Observing that $\lambda_i = \mu_i^2$, where $\mu_1 = 1 + \sqrt{2}$ and $\mu_2 = 1 - \sqrt{2}$, we see

$$\begin{aligned} d_n &= \gcd\left(\frac{\lambda_1^n + \lambda_2^n}{2} - 1, \frac{\lambda_1^n - \lambda_2^n}{2\sqrt{2}}\right) \\ &= \gcd\left(\frac{(\mu_1^n - \mu_2^n)^2}{2}, \frac{(\mu_1^n - \mu_2^n)(\mu_1^n + \mu_2^n)}{2\sqrt{2}}\right) \\ &= \left(\frac{\mu_1^n - \mu_2^n}{\sqrt{2}}\right)\gcd\left(\frac{\mu_1^n - \mu_2^n}{\sqrt{2}}, \frac{\mu_1^n + \mu_2^n}{2}\right) \end{aligned}$$

since $(\mu_1^n - \mu_2^n)/\sqrt{2}$, and $(\mu_1^n + \mu_2^n)/2$ are rational integers. As $|\mu_1| > 1$ and $|\mu_2| < 1$, we conclude $\lim_{n\to\infty}(\mu_1^n - \mu_2^n) = \infty$. Hence, $\lim_{n\to\infty} d_n = \infty$.

Comment: The proof extends to establishing the same result for integral matrices $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of determinant 1 and $|\text{trace}(A)| > 1$ (the latter to guarantee $r_n \to \infty$ where $r_n = \text{trace}(A)r_{n-1} - r_{n-2}$). A similar argument gives the same conclusion for the entries of $A^n - I$.

B-5 (11, 4, 4, 0, 0, 0, 0, 0, 32, 10, 15, 130)

Solution. For any $\alpha \geq 0$ and any positive integer k, we have

$$f_\alpha^k(n^2) = \lfloor \alpha \lfloor \alpha \lfloor \cdots \lfloor \alpha n^2 \rfloor \cdots \rfloor \rfloor \rfloor \leq \lfloor \alpha \cdot \alpha \cdots \alpha n^2 \rfloor = f_{\alpha^k}(n^2),$$

so it is enough to show that there exists an $\alpha \geq 0$ such that for $1 \leq k \leq n$,

$$f_\alpha^k(n^2) \geq n^2 - k \quad \text{and} \quad \alpha^k n^2 < n^2 - k + 1.$$

For $k = 1$, the first of these two inequalities yields $\alpha n^2 \geq n^2 - 1$; we will show that $\alpha = (n^2 - 1)/n^2 = 1 - 1/n^2$ will do. Using this value of α, we use induction on k to show that $f_\alpha^k(n^2) \geq n^2 - k$ for $1 \leq k \leq n$; in fact, if this holds for k, we have

$$\begin{aligned}
f_\alpha^{k+1}(n^2) &\geq f_\alpha(n^2-k) \\
&\left\lfloor \left(1-\frac{1}{n^2}\right)(n^2-k) \right\rfloor \\
&\geq \left\lfloor \left(1-\frac{1}{n^2-k}\right)(n^2-k) \right\rfloor \\
&= n^2-(k+1)
\end{aligned}$$

completing the induction.

To show that $\alpha^k n^2 < n^2 - k + 1$, note that this inequality is clear when $n = 1$ and hence $k = 1$, $\alpha = 0$; for $n > 1$, the inequality is equivalent to

$$\alpha^k < 1 - \frac{k-1}{n^2}$$

$$\left(\frac{n^2-1}{n^2}\right)^k < 1 - \frac{k-1}{n^2},$$

$$\left(\frac{n^2}{n^2-1}\right)^k > \frac{1}{1-\dfrac{k-1}{n^2}},$$

$$\left(\frac{n^2}{n^2-1}\right)^{k-1} < \frac{n^2-1}{n^2} \cdot \frac{1}{1-\dfrac{k-1}{n^2}} = \frac{n^2-1}{n^2-k+1}.$$

Now,

$$\left(\frac{n^2}{n^2-1}\right)^{k-1} = \left(1+\frac{1}{n^2-1}\right)^{k-1} \geq 1 + \frac{k-1}{n^2-1} = \frac{n^2+k-2}{n^2-1},$$

and it is easy to see by cross-multiplication that for $1 \leq k \leq n$,

$$\frac{n^2+k-2}{n^2-1} > \frac{n^2-1}{n^2-k+1},$$

completing the proof.

B-6 (14, 11, 1, 0, 0, 0, 0, 0, 16, 10, 50, 104)

Solution. Observe that $n_a \equiv a \pmod{100}$ and $n_a \equiv 2^a \pmod{101}$.

Suppose $n_a + n_b \equiv n_c + n_d \pmod{10100}$. Then $n_a + n_b \equiv n_c + n_d \pmod{101}$, so

$$2^a + 2^b \equiv 2^c + 2^d \pmod{101}. \tag{1}$$

Also, $n_a + n_b \equiv n_c + n_d \pmod{100}$, so $a + b \equiv c + d \pmod{100}$, and therefore, by Fermat's Theorem (since 101 is prime), $2^{a+b} \equiv 2^{c+d} \pmod{101}$. That is,

$$2^a \cdot 2^b \equiv 2^c \cdot 2^d \pmod{101}. \tag{2}$$

From (1) and (2), we see that $\{2^a, 2^b\}$ and $\{2^c, 2^d\}$ are the same set modulo 101, namely, the set of roots of the quadratic polynomial $(x - 2^a)(x - 2^b) = x^2 - (2^a + 2^b)x + 2^a 2^b = (x - 2^c)(x - 2^d)$ in the field $\mathbf{Z}_{101}$. To see that $\{a, b\} = \{c, d\}$, it suffices to show that the numbers 2^a for $a \in \{0, 1, \ldots, 99\}$ are distincnt modulo 101. That is, we need to show that the order of 2 modulo 101 is precisely 100. For this, it suffices to show that $2^{20} \not\equiv 1 \pmod{101}$ and $2^{50} \not\equiv 1 \pmod{101}$. We have $2^{10} = 1024 \equiv 14 \pmod{101}$, so that $2^{20} \equiv 14^2 \equiv -6 \pmod{101}$, from which $2^{50} \equiv 2^{20} 2^{20} 2^{10} \equiv 36 \cdot 14 \equiv -1 \pmod{101}$.

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PICTURE PUZZLE
(from the collection of Paul Halmos)

Hint: He doesn't look the same now as he did in 1951.
(see page 690)