The Fifty-Sixth William Lowell Putnam Mathematical Competition

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The following are the results of the fifty-sixth William Lowell Putnam Mathematical Competition, held on December 2, 1995. They have been determined in accordance with the regulations governing the Competition, an annual contest supported by the William Lowell Putnam Prize Fund for the Promotion of Scholarship, a fund left by Mrs. Putnam in memory of her husband. The Competition is held under the auspices of the Mathematical Association of America.

The first prize, \$7,500, was awarded to the Department of Mathematics at Harvard University. The members of the winning team were Kiran S. Kedlaya, Lenhard L. Ng, and Hong Zhou; each was awarded a prize of \$500.

The second prize, \$5,000, was awarded to the Department of Mathematics at Cornell University. The members of the winning team were Jeremy L. Bem, Robert D. Kleinberg, and Mark Krosky; each was awarded a prize of \$400.

The third prize, \$3,000, was awarded to the Department of Mathematics at the Massachusetts Institute of Technology. The members of the winning team were Ruth A. Britto-Pacumio, Sergey M. Ioffe, and Thomas A. Weston; each was awarded a prize of \$300.

The fourth prize, \$2,000, was awarded to the Department of Mathematics at the University of Toronto. The members of the winning team were Edward Goldstein, J. P. Grossman, and Naoki Sato; each was awarded a prize of \$200.

The fifth prize, \$1,000, was awarded to the Department of Mathematics at Princeton University. The members of the winning team were Michael J. Goldberg, Alex Heneveld, and Jacob A. Rasmussen; each was awarded a prize of \$100.

The five highest ranking individual contestants, in alphabetical order, were Yevgeniy Dodis, New York University; J. P. Grossman, University of Toronto; Kiran S. Kedlaya, Harvard University; Sergey V. Levin, Harvard University; and Lenhard L. Ng, Harvard University. Each of these has been designated a Putnam Fellow by the Mathematical Association of America and awarded a prize of \$1,000 by the Putnam Prize Fund.

The next four highest ranking contestants, in alphabetical order, were Mark Krosky, Cornell University; Serban M. Nacu, Harvard University; Akira Negi, University of North Carolina, Chapel Hill; and Chung-Chieh Shan, Harvard University; each was awarded a prize of \$500.

The next five highest ranking contestants, in alphabetical order, were Aaron F. Archer, Harvey Mudd College; Jeremy L. Bem, Cornell University; Robert D. Kleinberg, Cornell University; David L. Savitt, University of British Columbia; and Hong Zhou, Harvard University; each was awarded a prize of \$250.

The next ten highest ranking individuals, in alphabetical order, are Federico Ardila, Massachusetts Institute of Technology; Robert G. Au, Stanford University; Ioana Dumitriu, New York University; Craig R. Helfgott, Princeton University; John J. Krueger, Hope College; Daniel K. Schepler, Washington University, St. Louis; Mikhail V. Shubov, Texas Tech University; Balint Virag, Harvard University; Ronald A. Walker, University of Richmond; and Stephen S. Wang, Harvard University; each was awarded a prize of $100.

The following teams, named in alphabetical order, received honorable mention; University of British Columbia, with team members Mark Hamilton, Erich Mueller, and David L. Savitt; University of Chicago, with team members Dean Jens, Christopher D. Jeris, and Jeffrey Willson; Duke University, with team members Johanna Miller, Noam Shazeer, and Tung Tran; New York University, with team members Yevgeniy Dodis, Ioana Dumitriu, and Yevgeniy Kovchegov; and Washington University, St. Louis, with team members Matthew Crawford, Daniel K. Schepler, and Jade P. Vinson.

Honorable mention was achieved by the following thirty-two individuals named in alphabetical order: Jared E. Anderson, University of Victoria; Donald A. Barkauskas, Rice University; William J. Beckler, Colgate University; Manjul Bhargava, Harvard University; David E. Bradley, Virginia Polytechnic Institute and State University; Ruth A. Britto-Pacumio, Massachusetts Institute of Technology; Samit Dasgupta, Harvard University; Todd W. Geldon, Princeton University; Andrei C. Gnepp, Harvard University; Michael J. Goldberg, Princeton University; Wei-Hwa Huang, California Institute of Technology; Sergey M. Ioffe, Massachusetts Institute of Technology; David Y. Jao, Massachusetts Institute of Technology; Christopher D. Jeris, University of Chicago; Sergey Kirshner, University of California, Berkeley; Mikhail G. Konikov, University of Maryland, College Park; Eric H. Kuo, Massachusetts Institute of Technology; Frédéric Latour, University of Waterloo; Dion Lew, University of Toronto; Adam W. Meyerson, Massachusetts Institute of Technology; Olexei I. Motrunich, University of Missouri, Columbia; Roman Muchnik, California Institute of Technology; Colin A. Percival, Simon Fraser University; Rajesh J. Pereira, McGill University; Jacob A. Rasmussen, Princeton University; Naoki Sato, University of Toronto; Douglas Squirrel, Reed College; Guido Ubaldis, University of California, Berkeley; Jade P. Vinson, Washington University, St. Louis; Jonathan L. Weinstein, Harvard University; Thomas A. Weston, Massachusetts Institute of Technology; and Liang Yang, Yale University.

The other individuals who achieved ranks among the top 102, in alphabetical order of their schools, were: University of British Columbia, Erich J. Mueller; University of California, Santa Barbara, Akshay Venkatesh; Carleton University, Bhaskara M. Marthi; University of Connecticut, William M. Watson; University of Delaware, Charles W. Helms; Duke University, Johanna L. Miller; Harvard University, Matthew L. Bruce, Hank S. Chien, Patrick K. Corn, Adam Kalai, Paul Li, David L. McAdams, Daniel S. Quint, Robert Ribciuc, Scott R. Sheffield, Florin Spinu, Eric G. Yeh; University of Illinois, Champaign-Urbana, Tsz Ho Chan; Massachusetts Institute of Technology, Matthew D. Blum, Amit Khetan, Sergei Krupenin, Alexander Morcos; McGill University, François Labelle; University of Missouri, Rolla, Hal J. Burch; City University of New York, Queen's College, Daniil Khaykis; Princeton University, Peter A. Coles, Alex Heneveld, Michael Krasnitz, Andrew M. Neitzke, Alexandru-Anton A. M. Popa, Ransom L. Richardson; Queen's University, Joanna Karczmarek; Reed College, Galen B. Huntington; Rice University, Ron D. Dror, Brian M. Wahlert; Stanford University, Christopher C. Cheng, Theodore H. Hwa; University of Toronto, Edward Goldstein, Cyrus Hsia, Alexander M. Nicholson; Vassar College, Andrew F. Rizzo; University of Victoria, Peter J. Dukes; University of Waterloo, Jeff W. Brown, Peter L. Milley, Wei Yu; and Whittier College, Joshua S. Worley.

The Elizabeth Lowell Putnam Prize, named for the wife of William Lowell Putnam and to be "awarded periodically to a woman whose performance on the Competition has been deemed particularly meritorious," is awarded this year to Ioana Dumitriu of New York University. The winner is awarded a prize of $500.

There were 2,468 individual contestants from the 405 colleges and universities in Canada and the United States in the competition of December 2, 1995. Teams were entered by 306 institutions. The Questions Committee for the fifty-sixth competition consisted of Fan Chung, University of Pennsylvania, chair; Mark I. Krusemeyer, Carleton College, and Richard K. Guy, University of Calgary; they composed the problems listed below and were most prominent among those suggesting solutions.

PROBLEMS

Problem A-1. Let S be a set of real numbers which is closed under multiplication (that is, if a and b are in S, then so is ab). Let T and U be disjoint subsets of S whose union is S. Given that the product of any *three* (not necessarily distinct) elements of T is in T and that the product of any three elements of U is in U, show that at least one of the two subsets T, U is closed under multiplication.

Problem A-2. For what pairs (a, b) of positive real numbers does the improper integral

$$\int_b^{\infty}\left(\sqrt{\sqrt{x+a}-\sqrt{x}}-\sqrt{\sqrt{x}-\sqrt{x-b}}\right)dx$$

converge?

Problem A-3. The number $d_1d_2\dots d_9$ has nine (not necessarily distinct) decimal digits. The number $e_1e_2\dots e_9$ is such that each of the nine 9-digit numbers formed by replacing just one of the digits d_i in $d_1d_2\dots d_9$ by the corresponding digit e_i $(1 \le i \le 9)$ is divisible by 7. The number $f_1f_2\dots f_9$ is related to $e_1e_2\dots e_9$ in the same way: that is, each of the nine numbers formed by replacing one of the e_i by the corresponding f_i is divisible by 7. Show that, for each i, $d_i - f_i$ is divisible by 7.

[For example, if $d_1d_2\dots d_9 = 199501996$, then e_6 may be 2 or 9, since 199502996 and 199509996 are multiples of 7.]

Problem A-4. Suppose we have a necklace of n beads. Each bead is labeled with an integer and the sum of all these labels is $n-1$. Prove that we can cut the necklace to form a string whose consecutive labels $x_1, x_2, \dots, x_n$ satisfy

$$\sum_{i=1}^{k} x_i \le k-1 \qquad \text{for} \quad k = 1, 2, \dots, n.$$

Problem A-5. Let $x_1, x_2, \ldots, x_n$ be differentiable (real-valued) functions of a single variable t which satisfy

$$\begin{aligned}
\frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
\frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
&\vdots \qquad \vdots \qquad \vdots \\
\frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n
\end{aligned}$$

for some constants $a_{ij} \geq 0$. Suppose that for all i, $x_i(t) \to 0$ as $t \to \infty$. Are the functions $x_1, x_2, \ldots, x_n$ necessarily linearly dependent?

Problem A-6. Suppose that each of n people writes down the numbers $1, 2, 3$ in random order in one column of a $3 \times n$ matrix, with all orders equally likely and with the orders for different columns independent of each other. Let the row sums a, b, c of the resulting matrix be rearranged (if necessary) so that $a \leq b \leq c$. Show that for some $n \geq 1995$, it is at least four times as likely that both $b = a + 1$ and $c = a + 2$ as that $a = b = c$.

Problem B-1. For a partition π of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, let $\pi(x)$ be the number of elements in the part containing x. Prove that for any two partitions π and π', there are two distinct numbers x and y in $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ such that $\pi(x) = \pi(y)$ and $\pi'(x) = \pi'(y)$. [A *partition* of a set S is a collection of disjoint subsets (parts) whose union is S.]

Problem B-2. An ellipse, whose semi-axes have lengths a and b, rolls without slipping on the curve $y = c\sin(x/a)$. How are a, b, c related, given that the ellipse completes one revolution when it traverses one period of the curve?

Problem B-3. To each positive integer with n^2 decimal digits we associate the determinant of the matrix obtained by writing the digits in order across the rows. For example, for $n = 2$, to the integer 8617 we associate $\det\begin{pmatrix} 8 & 6 \\ 1 & 7 \end{pmatrix} = 50$. Find, as a function of n, the sum of all the determinants associated with n^2-digit integers. (Leading digits are assumed to be nonzero; for example, for $n = 2$, there are 9000 determinants.)

Problem B-4. Evaluate

$$\sqrt[8]{2207 - \cfrac{1}{2207 - \cfrac{1}{2207 - \cdots}}}.$$

Express your answer in the form $(a + b\sqrt{c})/d$, where a, b, c, d are integers.

Problem B-5. A game starts with four heaps of beans, containing 3, 4, 5 and 6 beans. The two players move alternately. A move consists of taking *either*

a. one bean from a heap, provided at least two beans are left behind in that heap, or
b. a complete heap of two or three beans.

The player who takes the last heap wins. To win the game, do you want to move first or second? Give a winning strategy.

Problem B-6. For a positive real number α, define

$$S(\alpha) = \{\lfloor n\alpha \rfloor : n = 1, 2, 3, \dots\}.$$

Prove that $\{1, 2, 3, \dots\}$ cannot be expressed as the disjoint union of three sets $S(\alpha)$, $S(\beta)$, and $S(\gamma)$. [As usual, $\lfloor x \rfloor$ is the greatest integer $\le x$.]

SOLUTIONS. In the 12-tuples $(n_{10}, n_9, n_8, n_7, n_6, n_5, n_4, n_3, n_2, n_1, n_0, n_{-1})$ following each problem number below, n_i for $10 \ge i \ge 0$ is the number of students among the top 204 contestants achieving i points for the problem and n_{-1} is the number of those not submitting solutions.

A-1 (115, 64, 20, 0, 0, 0, 0, 0, 1, 0, 2, 2)

Solution. Suppose T is not closed under multiplication. Then there are elements $t_1, t_2 \in T$ with $t_1 t_2 \notin T$, and since S is closed under multiplication, $t_1 t_2 \in U$. Now consider any two elements $u_1, u_2 \in U$; we'll show that $u_1 u_2 \in U$ (and thus that U is closed under multiplication). Suppose $u_1 u_2 \notin U$. Then $u_1 u_2 \in T$, so $t_1 \cdot t_2 \cdot u_1 u_2 \in T$ (as a product of three elements of T), but also $t_1 \cdot t_2 \cdot u_1 u_2 = (t_1 t_2) \cdot u_1 \cdot u_2 \in U$ (as a product of three elements of U), which is a contradiction since T and U are disjoint. So $u_1 u_2 \in U$, and we are done.

Comment. T, U need not both be closed; for example, if T (respectively U) is the set of positive integers $\equiv 1$ (respectively 3) modulo 4, then $S = T \cup U$ and T are closed, but U is not.

A-2 (26, 10, 14, 0, 0, 0, 0, 0, 14, 3, 56, 81)

Solution 1. The integral converges if and only if $a = b$.

Note that the integrand is defined and continuous for all $x \ge b$, so the only issue is convergence at ∞.

Since $(\sqrt{x+a} - \sqrt{x})(\sqrt{x+a} + \sqrt{x}) = a$ and $(\sqrt{x} - \sqrt{x-b})(\sqrt{x} + \sqrt{x-b}) = b$, we can rewrite the integrand as

$$\frac{\sqrt{a}}{\sqrt{\sqrt{x+a} + \sqrt{x}}} - \frac{\sqrt{b}}{\sqrt{\sqrt{x} + \sqrt{x-b}}}$$

$$= \frac{1}{\sqrt[4]{x}} \left[\frac{\sqrt{a}}{\sqrt{\sqrt{1 + \dfrac{a}{x}} + 1}} - \frac{\sqrt{b}}{\sqrt{1 + \sqrt{1 - \dfrac{b}{x}}}} \right].$$

If $a \neq b$, the quantity in the brackets approaches the nonzero number $\sqrt{a/2} - \sqrt{b/2}$ as $x \to \infty$, so for large enough x the absolute value of the integrand is at least $c/2\sqrt[4]{x}$, where $c = |\sqrt{a/2} - \sqrt{b/2}|$. The integrand then diverges by comparison with the divergent integral $\int_1^\infty (1/\sqrt[4]{x})\, dx$.

On the other hand, if $a = b$, the integrand equals

$$\frac{\sqrt{a}}{\sqrt[4]{x}}\left(\frac{1}{\sqrt{\sqrt{1+\frac{a}{x}}+1}} - \frac{1}{\sqrt{1+\sqrt{1-\frac{a}{x}}}}\right)$$

$$= \frac{\sqrt{a}}{\sqrt[4]{x}} \cdot \frac{1}{\sqrt{\sqrt{1+\frac{a}{x}}+1}\sqrt{1+\sqrt{1-\frac{a}{x}}}}\left(\sqrt{1+\sqrt{1-\frac{a}{x}}} - \sqrt{\sqrt{1+\frac{a}{x}}+1}\right)$$

$$= \frac{\sqrt{a}}{\sqrt[4]{x}} \cdot \frac{1}{\sqrt{\sqrt{1+\frac{a}{x}}+1}\sqrt{1+\sqrt{1-\frac{a}{x}}}} \frac{1+\sqrt{1-\frac{a}{x}} - \left(\sqrt{1+\frac{a}{x}}+1\right)}{\sqrt{1+\sqrt{1-\frac{a}{x}}} + \sqrt{\sqrt{1+\frac{a}{x}}+1}}$$

$$= \frac{\sqrt{a}}{\sqrt[4]{x}} \cdot \frac{1}{\varphi(x)}\left(\sqrt{1-\frac{a}{x}} - \sqrt{1+\frac{a}{x}}\right)$$

$$= \frac{\sqrt{a}}{\sqrt[4]{x}} \cdot \frac{1}{\varphi(x)} \cdot \frac{1-\frac{a}{x} - \left(1+\frac{a}{x}\right)}{\sqrt{1-\frac{a}{x}} + \sqrt{1+\frac{a}{x}}} = \frac{-2a\sqrt{a}}{x\sqrt[4]{x}\,\varphi(x)\left(\sqrt{1-\frac{a}{x}} + \sqrt{1+\frac{a}{x}}\right)},$$

where

$$\varphi(x) = \sqrt{\sqrt{1+\frac{a}{x}}+1}\sqrt{1+\sqrt{1-\frac{a}{x}}}\left(\sqrt{1+\sqrt{1-\frac{a}{x}}} + \sqrt{\sqrt{1+\frac{a}{x}}+1}\right).$$

Since $\varphi(x) \to \sqrt{2} \cdot \sqrt{2} \cdot (\sqrt{2} + \sqrt{2}) = 4\sqrt{2}$ as $x \to \infty$, the absolute value of the integrand is then less than $a\sqrt{a}/x^{5/4}$, and the integral converges by comparison with $\int_1^\infty (1/x^{5/4})\,dx$.

Solution 2. The integrand equals

$$x^{1/4}\left(\sqrt{\left(1+\frac{a}{x}\right)^{1/2} - 1} - \sqrt{1-\left(1-\frac{b}{x}\right)^{1/2}}\right)$$

$$= x^{1/4}\left(\sqrt{\frac{a}{2x} - \frac{a^2}{8x^2} + O(x^{-3})} - \sqrt{\frac{b}{2x} + \frac{b^2}{8x^2} + O(x^{-3})}\right)$$

$$= x^{-1/4}\left(\sqrt{\frac{a}{2}}\sqrt{1-\frac{a}{4x}+O(x^{-2})} - \sqrt{\frac{b}{2}}\sqrt{1+\frac{b}{4x}+O(x^{-2})}\right)$$

$$= x^{-1/4}\left(\sqrt{\frac{a}{2}}\left[1-\frac{a}{8x}+O(x^{-2})\right] - \sqrt{\frac{b}{2}}\left[1+\frac{b}{8x}+O(x^{-2})\right]\right)$$

$$= x^{-1/4}\left(\sqrt{\frac{a}{2}} - \sqrt{\frac{b}{2}}\right) + O(x^{-5/4}),$$

so since $\int_1^\infty x^{-1/4}\,dx$ diverges and $\int_1^\infty x^{-5/4}\,dx$ converges, the integral converges just if $a = b$.

Solution 3. Let $f(x) = \sqrt{x}$. By the Mean Value Theorem there is a number h (which depends on x), $0 < h < a$, such that $f(x+a) - f(x) = f'(x+h)a$, and a number k (which depends on x), $0 < k < b$, such that $f(x) - f(x-b) = f'(x-k)b$. Thus,

$$\sqrt{x+a} - \sqrt{x} = \frac{a}{2\sqrt{x+h}} \qquad \text{for some } h,\ 0 < h < a, \quad \text{and}$$

$$\sqrt{x} - \sqrt{x-b} = \frac{b}{2\sqrt{x-k}} \qquad \text{for some } k,\ 0 < k < b.$$

It follows that

$$\sqrt{\sqrt{x+a} - \sqrt{x}} - \sqrt{\sqrt{x} - \sqrt{x-b}}$$

$$= \frac{\sqrt{a}}{\sqrt{2}\,(x+h)^{1/4}} - \frac{\sqrt{b}}{\sqrt{2}\,(x-k)^{1/4}}$$

$$= \frac{(x-k)^{1/4}\sqrt{a} - (x+h)^{1/4}\sqrt{b}}{\sqrt{2}\,(x+h)^{1/4}(x-k)^{1/4}}$$

$$= \frac{(a^2-b^2)x - (ka^2+hb^2)}{\sqrt{2}\,(x+h)^{1/4}(x-k)^{1/4}\left[(x-k)^{1/4}\sqrt{a} + (x+h)^{1/4}\sqrt{b}\right]\left[(x-k)^{1/2}a + (x+h)^{1/2}b\right]}.$$

Thus, the integrand is $O(x^{-1/4})$ when $a \neq b$ and $O(x^{-5/4})$ when $a = b$. The result follows as in the previous solutions.

A-3 (95, 44, 39, 0, 0, 0, 0, 0, 12, 5, 3, 6)

Solution. Suppose $d_1 d_2 \ldots d_9 \equiv a \pmod 7$. Then for each i, $1 \le i \le 9$,

$$a \equiv a - 0 \equiv (d_1 \ldots d_i \ldots d_9) - (d_1 \ldots e_i \ldots d_9) \pmod 7$$
$$= 10^{9-i} d_i - 10^{9-i} e_i \pmod 7.$$

On summing these congruences, we find that $9a \equiv (d_1 d_2 \ldots d_9) - (e_1 e_2 \ldots e_9) \pmod 7$, and therefore $e_1 e_2 \ldots e_9 \equiv -a \pmod 7$.

In a similar manner, starting with $e_1 e_2 \ldots e_9 \equiv -a \pmod 7$, we have

$$-a \equiv 10^{9-i} e_i - 10^{9-i} f_i \pmod 7,$$

and therefore

$$-a \equiv \left(10^{9-i} d_i - a\right) - 10^{9-i} f_i \pmod 7,$$

or equivalently,

$$10^{9-i} d_i \equiv 10^{9-i} f_i \pmod 7.$$

Since 7 and 10 are relatively prime, $d_i \equiv f_i \pmod 7$.

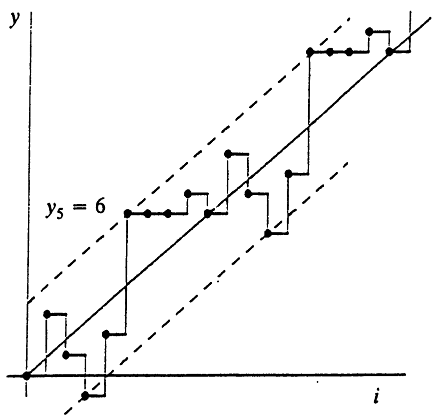
A-4 (39, 9, 13, 0, 0, 0, 0, 0, 19, 1, 83, 40)

Solution 1. We will show that there are just two places where we may cut the necklace. Each is associated with the sense in which we go around the necklace.

Choose an arbitrary starting position, and a sense of rotation, and let the labels be the integers $y_1, y_2, \ldots, y_n$, where $\Sigma_{i=1}^n y_i = n - 1$.

Consider the path in the coordinate plane which starts from the origin, (0, 0), moves one space to the right and then vertically to the point $(1, y_1)$, then one space to the right and vertically to the point $(2, y_1 + y_2)$, and so on to the points $(3, y_1 + y_2 + y_3), \ldots, (k, \Sigma_{i=1}^{k} y_i), \ldots, (n, \Sigma_{i=1}^{n} y_i = n - 1)$. Continue on around the necklace, repeating the pattern: $(n + 1, n - 1 + y_1), (n + 2, n - 1 + y_1 + y_2), \ldots$. Choose the point(s) $(K, \Sigma_{i=1}^{K} y_i)$ which maximize the height above the line $y = ((n - 1)/n)x$; that is, maximize $\Sigma_{i=1}^{k} y_i - ((n - 1)/n)k$. Since $n - 1$ and n are relatively prime integers, K is unique modulo n. Relabel the integers, decreasing the subscripts by K, that is, move the origin to the chosen point. Since the slope of the line from this point to any other is at most $(n - 1)/n$ (with equality only at the end of each period), we have achieved our aim: if $\Sigma_{i=1}^{k} x_i \geq k$, the slope would be at least 1.

Example. Suppose $\{y_1, y_2, \ldots, y_n\} = \{3, -2, -2, 3, 6, 0, 0, 1, -1\}$. The corresponding path is shown below.



The shifted sequence $0, 0, 1, -1, 3, -2, -2, 3, 6$ has partial sums $0, 0, 1, 0, 3, 1, -1, 2, 8 \leq k - 1 = 0, 1, 2, 3, 4, 5, 6, 7, 8$.

To find the cutting place associated with the opposite sense, we don't need to redraw the graph: simply select the *lowest* point below the line to get the shifted sequence $-2, -2, 3, -1, 1, 0, 0, 6, 3$ with partial sums $-2, -4, -1, -2, -1, -1, -1, 5, 8 \leq 0, 1, 2, 3, 4, 5, 6, 7, 8$.

Solution 2. Let the labels be denoted by $y_1, y_2, \ldots, y_n$, and suppose that there is no such k satisfying the above statement. Then for every k, there is a least number $\alpha_k \geq 1$ such that

$$\sum_{i=k}^{k-1+\alpha_k} y_i \geq \alpha_k,$$

where the subscripts i are taken modulo n. (Note that there are α_k terms in the sum.) We may assume that the labels are given so that $\alpha_1 \geq \alpha_i$ for $i = 1, 2, 3, \ldots, n$. We now choose numbers k_i as follows.

First, define $k_1 = 1$ and let β_1 be the smallest integer such that

$$\sum_{i=1}^{\beta_1} y_i \geq \beta_1.$$

(In this case, $\beta_1 = \alpha_1$.) Clearly $\beta_1 < n$. Thus, $I_1 = \{1, 2, \ldots, \beta_1\}$ is a proper subset of $\{1, 2, \ldots, n\}$.

Suppose that we have defined k_j, β_j, and I_j for $j < i$, with $k_1 < k_2 < \cdots < k_{i-1}$, and suppose we know that the I_j are disjoint sets whose union is $\{1, 2, \dots, k_{i-1} - 1 + \beta_{i-1}\}$ and that this union is properly contained in $\{1, 2, \dots, n\}$. Proceed as follows.

Define $k_i = k_{i-1} + \beta_{i-1}$, and choose β_i to be the smallest integer such that

$$\sum_{j=k_i}^{k_i-1+\beta_i} y_j \geq \beta_i.$$

(Note that $\beta_i = \alpha_{k_i}$.) Let $I_i = \{k_i, k_i + 1, \dots, k_i - 1 + \beta_i\}$, and observe that I_i has β_i elements. We claim that $k_i - 1 + \beta_i < n$.

Case 1. Suppose that $k_i - 1 + \beta_i = n$. In this case,

$$\begin{aligned} n - 1 = \sum_{j=1}^{n} y_j &= \sum_{j=1}^{i-1} \left(\sum_{r=k_j}^{k_{j+1}-1} y_r \right) + \sum_{j=k_i}^{k_i-1+\beta_i} y_j \geq \sum_{j=1}^{i-1} \beta_j + \beta_i \\ &= \sum_{j=1}^{i-1} (k_{j+1} - k_j) + \beta_i = n, \end{aligned}$$

a contradiction.

Case 2. Suppose that $k_i - 1 + \beta_i > n$. Let $s = n + 1 - k_i$ and $t = k_i - 1 + \beta_i - n$ (note that $\beta_i = s + t$) and suppose that $t < \beta_1$. Then

$$\beta_i \leq \sum_{j=k_i}^{k_i-1+\beta_i} y_j = \sum_{j=k_i}^{k_i-1+s} y_j + \sum_{j=1}^{t} y_j \leq (s - 1) + (t - 1) = \beta_i - 2,$$

which is a contradiction. Therefore $t \geq \beta_1$ so that $\beta_i > \beta_1$. But this is a contradiction because $\beta_i = \alpha_{k_i} \leq \alpha_1 = \beta_1$.

We conclude that $k_i - 1 + \beta_i < n$, and so the I_i can never cover the entire set $\{1, 2, \dots, n\}$, which is clearly absurd.

Therefore the necklace can be cut in the desired manner.

A-5 (1, 1, 2, 1, 0, 0, 0, 6, 18, 14, 43, 118)

Solution. We will show that the functions $x_1, x_2, \dots, x_n$ are necessarily linearly dependent. Let

$$\mathbf{x} = (x_1, x_2, \dots, x_n) \quad \text{and} \quad \mathbf{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}.$$

Then we are given that $d\mathbf{x}/dt = \mathbf{A}\mathbf{x}$. Consider a linear combination $y = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$ of the given functions, where $\alpha_1, \alpha_2, \dots, \alpha_n$ are constants, possibly complex, to be chosen later. If we set $\mathbf{v} = (\alpha_1, \alpha_2, \dots, \alpha_n)$, we have $y = \mathbf{v} \cdot \mathbf{x} = \mathbf{v}^T \mathbf{x}$ (T = transpose) and thus

$$\frac{dy}{dt} = \mathbf{v}^T \frac{d\mathbf{x}}{dt} = \mathbf{v}^T \mathbf{A} \mathbf{x} = (\mathbf{A}^T \mathbf{v})^T \mathbf{x}.$$

In particular, if $\mathbf{v}$ is an eigenvector of $\mathbf{A}^T$ for the eigenvalue λ, we get

$$\frac{dy}{dt} = (\mathbf{A}^T \mathbf{v})^T \mathbf{x} = (\lambda \mathbf{v})^T \mathbf{x} = \lambda \mathbf{v} \cdot \mathbf{x} = \lambda y,$$

so in that case y has the form $y = Ce^{\lambda t}$ for some constant C.

Since we are given that $a_{ij} \geq 0$, in particular we have $\text{Trace}(\mathbf{A}^T) = a_{11} + a_{22} + \cdots + a_{nn} \geq 0$, so $\mathbf{A}^T$ has at least one eigenvalue whose real part is nonnegative. Let λ be such an eigenvalue, and let $\mathbf{v} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a corresponding eigenvector of $\mathbf{A}^T$. Then, by the above, we have $y = Ce^{\lambda t}$ with $\text{Re}(\lambda) \geq 0$. On the other hand, since $x_i(t) \to 0$ as $t \to \infty$ and y is a linear combination of the x_i, we have $y(t) \to 0$ as $t \to \infty$. But $|e^{\lambda t}| = e^{\text{Re}(\lambda)t} \geq 1$ for $t \geq 0$, so $Ce^{\lambda t} \to 0$ implies $C = 0$. Therefore, a nontrivial linear combination of the x_i is identically zero (note that $\mathbf{v} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is nonzero because it is an eigenvector), so the x_i are linearly dependent, and we are done.

A-6 (1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 61, 141)

Solution. For a positive integer n, and using the other notation of the problem, let $S(n)$ be the statement "it is at least four times as likely that both $b = a + 1$ and $c = a + 2$ as that $a = b = c$." We show that if $S(n)$ is false, then $S(n + 1)$ is true. In particular, $S(n)$ is true for at least one of $n = 1995$, $n = 1996$.

For any positive integer n, let X_n be the number of ways the matrix can be formed so that $b = a + 1$ and $c = a + 2$ (where a, b, c, with $a \leq b \leq c$, are the row sums after rearrangement; incidentally, $a + b + c = 6n$). Let Y_n be the number of ways the matrix can be formed so that $a = b = c$, and let Z_n be the number of ways with $a = b$ and $c = a + 3$.

Our assumption that $S(n)$ is false means that $4Y_n > X_n$. Now note that if a matrix *with* $n + 1$ *columns* is formed such that its row sums are all equal, then the first n columns of that matrix form one of the matrices that is counted by X_n. Conversely, for each of the matrices counted by X_n, there is exactly one way to "complete" it to a matrix counted by Y_{n+1}, so we have $Y_{n+1} = X_n$. Similar arguments show that $Z_{n+1} \geq X_n$ (since to row sums $a, a + 1, a + 2$ one can add $2, 1, 3$ respectively to get $a + 2, a + 2, a + 5$), and $X_{n+1} \geq 6Y_n + 2X_n + 2Z_n$ (since $a + 2, a + 3, a + 4$ can be obtained by adding $1, 2, 3$ *in any order* to $a + 1, a + 1, a + 1$ [and rearranging], or by adding $2, 3, 1$ or $3, 1, 2$ (in that order) to $a, a + 1, a + 2$ [and rearranging], or by adding $3, 2, 1$ or $2, 3, 1$ to $a, a, a + 3$ [and rearranging]).

Therefore, we have

$$\begin{aligned}
\frac{X_{n+1}}{Y_{n+1}} = \frac{X_{n+1}}{X_n} &\geq 6\frac{Y_n}{X_n} + 2 + 2\frac{Z_n}{X_n} \\
&\geq 6\frac{Y_n}{X_n} + 2 + 2\frac{X_{n-1}}{X_n} = 6\frac{Y_n}{X_n} + 2 + 2\frac{Y_n}{X_n} \\
&= 8\frac{Y_n}{X_n} + 2.
\end{aligned}$$

But by our assumption, we have $Y_n/X_n > 1/4$, so $X_{n+1}/Y_{n+1} \geq 8/4 + 2 = 4$, so $X_{n+1} \geq 4Y_{n+1}$, and we are done.

B-1 (124, 26, 7, 0, 0, 0, 0, 0, 4, 10, 11, 22)

Solution. Suppose there are no two such numbers in $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Then any two numbers in the same part of π must be contained in parts of different size in π', and vice versa. This implies that the largest parts of π and π' have at most three numbers in them (because $1 + 2 + 3 + 4 > 9$). In fact, any two numbers in parts of the same size in π must be contained in parts of different sizes in π'. Therefore, π can have at most one part of size 3, one part of size 2, and at most three parts of size 1. This is impossible for a partition of a set of 9 numbers.

B-2 (2, 54, 26, 0, 0, 0, 0, 0, 3, 13, 51, 55)

Solution. We shall show that $b^2 = a^2 + c^2$.

The ellipse is given parametrically by the equations $x = a\cos\theta,\ y = b\sin\theta$. Using the familiar formula for arc length, the perimeter of the ellipse is

$$\int_0^{2\pi}\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\,d\theta.$$

The length of one period of the sine curve is

$$\int_0^{2\pi a}\sqrt{1 + \frac{c^2}{a^2}\cos^2\left(\frac{x}{a}\right)}\,dx = \int_0^{2\pi}\sqrt{a^2 + c^2\cos^2\theta}\,d\theta.$$

Write $a^2 + c^2\cos^2\theta = a^2\sin^2\theta + (a^2 + c^2)\cos^2\theta$, and we see that the arc lengths will be equal if and only if $b^2 = a^2 + c^2$, as claimed.

B-3 (54, 15, 11, 0, 0, 0, 0, 0, 33, 15, 49, 27)

Solution. The sum is 45 for $n = 1$, 20250 for $n = 2$, and 0 for $n \geq 3$.

The case $n = 1$ is trivial: $1 + 2 + \cdots + 9 = 45$. Now let $n \geq 2$. Then for each $n \times n$ matrix with entries in $\{0, 1, 2, \ldots, 9\}$ there is another such matrix obtained by interchanging the last two columns. (If this matrix is equal to the original one, its determinant is zero.) Since interchanging two columns in a matrix changes its determinant to the opposite determinant, the sum of *all* determinants of matrices with entries in $\{0, 1, 2, \ldots, 9\}$ is zero. However, we are not supposed to take all such matrices, but only the ones that don't have a 0 in the upper left corner. If $n \geq 3$, interchanging the last two columns doesn't affect that corner, and so the required sum is 0 by the same argument. On the other hand, if $n = 2$, all determinants in the sum cancel *except* those of the form $\det\begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$. These determinants only depend on the diagonal entries, and there are ten of them for each pair of diagonal entries; thus their sum is

$$10\sum_{i,j=1}^{9} ij = 10\left(\sum_{i=1}^{9} i\right)\left(\sum_{j=1}^{9} j\right) = 10 \cdot 45 \cdot 45 = 20250,$$

as claimed.

B-4 (9, 18, 62, 4, 0, 0, 0, 2, 21, 33, 10, 45)

Solution. The answer is $(3 + \sqrt{5})/2$.

Let

$$F(n) = n - \cfrac{1}{n - \cfrac{1}{n - \cdots}},$$

so the problem asks for $\sqrt[8]{F(2207)}$. Note that $F(n) = n - 1/F(n)$, and solving this quadratic equation for $F(n)$ yields $F(n) = (n \pm \sqrt{n^2 - 4})/2$.

For $n > 2$ we have

$$\frac{n - \sqrt{n^2 - 4}}{2} = \frac{2}{n + \sqrt{n^2 - 4}} < \frac{2}{n} < 1 < F(n),$$

so we must have the plus sign: $F(n) = (n + \sqrt{n^2 - 4})/2$.

Now note that

$$(F(n))^2 = \frac{n^2 + 2n\sqrt{n^2-4} + n^2 - 4}{4} = \frac{n^2 - 2 + n\sqrt{n^2-4}}{2}$$
$$= \frac{n^2 - 2 + \sqrt{(n^2-2)^2 - 4}}{2} = F(n^2 - 2)$$

for $n > 2$, since $n > 2$ implies $n^2 - 2 > 2$.

Conversely, if $k > 2$, then we have $k = n^2 - 2$ with $n = \sqrt{k+2} > 2$, and therefore $F(k) = (F(n))^2$, $\sqrt{F(k)} = F(n) = F(\sqrt{k+2})$. In particular,

$$\sqrt{F(2207)} = F(\sqrt{2209}) = F(47),$$
$$\sqrt[4]{F(2207)} = \sqrt{F(47)} = F(\sqrt{49}) = F(7), \quad \text{and}$$
$$\sqrt[8]{F(2207)} = \sqrt{F(7)} = F(\sqrt{9}) = F(3) = \frac{3+\sqrt{5}}{2}.$$

B-5 (72, 11, 13, 0, 0, 0, 0, 0, 9, 6, 42, 51)

Solution. Heaps of 0 or 1 cannot affect the game. In fact, heaps of 1 cannot arise. Heaps of 2 behave as though they were a single bean which may be removed. Heaps of 3 are special. Otherwise a move just removes a bean, and the result depends only on the parity of the total number of beans (counting a heap of 2 as a single bean).

The first player wins by taking one bean from the 3-heap, leaving heaps of 2, 4, 5 and 6 beans, whose "sum" is 1(= 2) + 4 + 5 + 6 which is even. Now the win is automatic, since the opponent must make the "sum" odd. It doesn't matter what moves are made, *except* that the first player mustn't move in a 4-heap (there is no need to since the sum will always be odd, and all the heaps can't be 4-heaps), and whenever the second player moves in a 4-heap, the first player removes all the remaining beans at the next move.

B-6 (3, 2, 0, 0, 0, 0, 0, 0, 2, 3, 61, 133)

Solution. Suppose $\alpha < \beta < \gamma$ and $S(\alpha)$, $S(\beta)$, and $S(\gamma)$ disjointly cover $\{1, 2, 3, \dots\}$. Since $\lfloor \alpha \rfloor = 1$, we have $\alpha = 1 + \epsilon$, for some ϵ satisfying $0 \le \epsilon < 1$.

Let $r > 1$ be the first value not in $S(\alpha)$. We have

$$\lfloor (r-1)\alpha \rfloor = r - 1, \quad \lfloor r\alpha \rfloor = r + 1.$$

Therefore,

$$(r-1)\alpha < r, \quad r\alpha \ge r + 1$$

and

$$1 + \frac{1}{r} \le \alpha < 1 + \frac{1}{r-1};$$

that is,

$$\frac{1}{r} \le \epsilon < \frac{1}{r-1}.$$

Fact 1. *If $u \notin S(\alpha)$, then the next element missing from $S(\alpha)$ is either $u + r$ or $u + r + 1$ (and the other of $u + r, u + r + 1$ is in $S(\alpha)$).*

Proof: Suppose $\lfloor t\alpha \rfloor = u - 1$, $\lfloor (t+1)\alpha \rfloor = u + 1$. Let $\delta = (t+1)\alpha - (u+1)$. The next missing element occurs at $u + m$ where m is the smallest integer such that $\delta + (m-1)\epsilon \geq 1$. If $m \leq r - 1$, we have

$$\delta + (m-1)\epsilon < m\epsilon \leq (r-1)\epsilon < 1$$

since $\delta < \epsilon$. Also, for $m = r + 1$,

$$\delta + (m-1)\epsilon = \delta + r\epsilon \geq 1.$$

Therefore $m = r$ or $r + 1$.

Note that $\lfloor \beta \rfloor = r$, so we have $r \leq \beta < r + 1$.

Fact 2. *If $v \in S(\beta)$, then the next element in $S(\beta)$ is $v + r$ or $v + r + 1$.*

Fact 2 can be proved in the same manner as Fact 1.

Combining Facts 1 and 2 with the fact that $S(\alpha)$ and $S(\beta)$ are disjoint, we conclude that the union of $S(\alpha)$ and $S(\beta)$ is all of $\{1, 2, 3, \dots\}$. Therefore, $\{1, 2, 3, \dots\}$ cannot be expressed as the disjoint union of *three* sets $S(\alpha)$, $S(\beta)$ and $S(\gamma)$.

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$\lim_{x \to \infty}(x^n/e^x) = 0$ for each $n = 1, 2, \dots$.

Because $e^x > x$ for all $x > 0$, the function $f(x) \equiv x/e^x$ is bounded on $(0, \infty)$. Then $g_n(x) \equiv x^n/e^x = n^n(f(x/n))^n$ is bounded on $(0, \infty)$ for each $n = 1, 2, \dots$ and $0 \leq x^n/e^x = g_{n+1}(x)/x \to 0$ as $x \to \infty$.

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