The Fifty-Seventh Annual William Lowell Putnam Mathematical Competition
Saturday, December 7, 1996

A-1 Find the least number $A$ such that for any two squares of combined area 1, a rectangle of area $A$ exists such that the two squares can be packed in the rectangle (without interior overlap). You may assume that the sides of the squares are parallel to the sides of the rectangle.

A-2 Let $C_1$ and $C_2$ be circles whose centers are 10 units apart, and whose radii are 1 and 3. Find, with proof, the locus of all points $M$ for which there exists points $X$ on $C_1$ and $Y$ on $C_2$ such that $M$ is the midpoint of the line segment $XY$.

A-3 Suppose that each of 20 students has made a choice of anywhere from 0 to 6 courses from a total of 6 courses offered. Prove or disprove: there are 5 students and 2 courses such that all 5 have chosen both courses or all 5 have chosen neither course.

A-4 Let $S$ be the set of ordered triples $(a, b, c)$ of distinct elements of a finite set $A$. Suppose that

1. $(a, b, c) \in S$ if and only if $(b, c, a) \in S$;
2. $(a, b, c) \in S$ if and only if $(c, b, a) \notin S$;
3. $(a, b, c)$ and $(c, d, a)$ are both in $S$ if and only if $(b, c, d)$ and $(d, a, b)$ are both in $S$.

Prove that there exists a one-to-one function $g$ from $A$ to $R$ such that $g(a) < g(b) < g(c)$ implies $(a, b, c) \in S$. Note: $R$ is the set of real numbers.

A-5 If $p$ is a prime number greater than 3 and $k = \lfloor 2p/3 \rfloor$, prove that the sum

$$\binom{p}{1} + \binom{p}{2} + \cdots + \binom{p}{k}$$

of binomial coefficients is divisible by $p^2$.

A-6 Let $c > 0$ be a constant. Give a complete description, with proof, of the set of all continuous functions $f : R \to R$ such that $f(x) = f(x^2 + c)$ for all $x \in R$. Note that $R$ denotes the set of real numbers.

B-1 Define a **selfish** set to be a set which has its own cardinality (number of elements) as an element. Find, with proof, the number of subsets of $\{1, 2, \ldots, n\}$ which are *minimal* selfish sets, that is, selfish sets none of whose proper subsets is selfish.

B-2 Show that for every positive integer $n$,

$$\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2}} < 1 \cdot 3 \cdot 5 \cdots (2n-1) < \left(\frac{2n+1}{e}\right)^{\frac{2n+1}{2}}.$$

B-3 Given that $\{x_1, x_2, \ldots, x_n\} = \{1, 2, \ldots, n\}$, find, with proof, the largest possible value, as a function of $n$ (with $n \geq 2$), of

$$x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1.$$

B-4 For any square matrix $A$, we can define $\sin A$ by the usual power series:

$$\sin A = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} A^{2n+1}.$$

Prove or disprove: there exists a $2 \times 2$ matrix $A$ with real entries such that

$$\sin A = \begin{pmatrix} 1 & 1996 \\ 0 & 1 \end{pmatrix}.$$

B-5 Given a finite string $S$ of symbols $X$ and $O$, we write $\Delta(S)$ for the number of $X$'s in $S$ minus the number of $O$'s. For example, $\Delta(XOOXOOX) = -1$. We call a string $S$ **balanced** if every substring $T$ of (consecutive symbols of) $S$ has $-2 \leq \Delta(T) \leq 2$. Thus, $XOOXOOX$ is not balanced, since it contains the substring $OOXOO$. Find, with proof, the number of balanced strings of length $n$.

B-6 Let $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$ be the vertices of a convex polygon which contains the origin in its interior. Prove that there exist positive real numbers $x$ and $y$ such that

$$(a_1, b_1)x^{a_1}y^{b_1} + (a_2, b_2)x^{a_2}y^{b_2} + \cdots + (a_n, b_n)x^{a_n}y^{b_n} = (0, 0).$$

**Solutions to the Fifty-Seventh William Lowell Putnam Mathematical Competition**

Saturday, December 7, 1996

Prepared by Manjul Bhargava and Kiran S. Kedlaya

A-1 If $x$ and $y$ are the sides of two squares with combined area 1, then $x^2+y^2=1$. Suppose without loss of generality that $x \geq y$. Then the shorter side of a rectangle containing both squares without overlap must be at least $x$, and the longer side must be at least $x+y$. Hence the desired value of $A$ is the maximum of $x(x+y)$.

To find this maximum, we let $x = \cos\theta, y = \sin\theta$ with $\theta \in [0, \pi/4]$. Then we are to maximize

$$\begin{aligned}
\cos^2\theta + \sin\theta\cos\theta &= \frac{1}{2}(1+\cos 2\theta + \sin 2\theta) \\
&= \frac{1}{2} + \frac{\sqrt{2}}{2}\cos(2\theta - \pi/4) \\
&\leq \frac{1+\sqrt{2}}{2},
\end{aligned}$$

with equality for $\theta = \pi/8$. Hence this value is the desired value of $A$.

A-2 Let $O_1$ and $O_2$ be the centers of $C_1$ and $C_2$, respectively. (We are assuming $C_1$ has radius 1 and $C_2$ has radius 3.) Then the desired locus is an annulus centered at the midpoint of $O_1O_2$, with inner radius 1 and outer radius 2.

For a fixed point $Q$ on $C_2$, the locus of the midpoints of the segments $PQ$ for $P$ lying on $C_1$ is the image of $C_1$ under a homothety centered at $Q$ of radius $1/2$, which is a circle of radius $1/2$. As $Q$ varies, the center of this smaller circle traces out a circle $C_3$ of radius $3/2$ (again by homothety). By considering the two positions of $Q$ on the line of centers of the circles, one sees that $C_3$ is centered at the midpoint of $O_1O_2$, and the locus is now clearly the specified annulus.

A-3 The claim is false. There are $\binom{6}{3} = 20$ ways to choose 3 of the 6 courses; have each student choose a different set of 3 courses. Then each pair of courses is chosen by 4 students (corresponding to the four ways to complete this pair to a set of 3 courses) and is not chosen by 4 students (corresponding to the 3-element subsets of the remaining 4 courses).

A-4 In fact, we will show that such a function $g$ exists with the property that $(a,b,c) \in S$ if and only if $g(d) < g(e) < g(f)$ for some cyclic permutation $(d,e,f)$ of $(a,b,c)$. We proceed by induction on the number of elements in $A$. If $A = \{a,b,c\}$ and $(a,b,c) \in S$, then choose $g$ with $g(a) < g(b) < g(c)$, otherwise choose $g$ with $g(a) > g(b) > g(c)$.

Now let $z$ be an element of $A$ and $B = A - \{z\}$. Let $a_1, \dots, a_n$ be the elements of $B$ labeled such that $g(a_1) < g(a_2) < \cdots < g(a_n)$. We claim that there exists a unique $i \in \{1, \dots, n\}$ such that $(a_i, z, a_{i+1}) \in S$, where hereafter $a_{n+k} = a_k$.

We show existence first. Suppose no such $i$ exists; then for all $i,k \in \{1,\dots,n\}$, we have $(a_{i+k}, z, a_i) \notin S$. This holds by property 1 for $k=1$ and by induction on $k$ in general, noting that

$$\begin{aligned}(a_{i+k+1}, z, a_{i+k}), (a_{i+k}, z, a_i) \in S &\Rightarrow (a_{i+k}, a_{i+k+1}, z), (z, a_i, a_{i+k}) \in S \\ &\Rightarrow (a_{i+k+1}, z, a_i) \in S.\end{aligned}$$

Applying this when $k = n$, we get $(a_{i-1}, z, a_i) \in S$, contradicting the fact that $(a_i, z, a_{i-1}) \in S$. Hence existence follows.

Now we show uniqueness. Suppose $(a_i, z, a_{i+1}) \in S$; then for any $j \neq i-1, i, i+1$, we have $(a_i, a_{i+1}, a_j), (a_j, a_{j+1}, a_i) \in S$ by the assumption on $G$. Therefore

$$\begin{aligned}(a_i, z, a_{i+1}), (a_{i+1}, a_j, a_i) \in S &\Rightarrow (a_j, a_i, z) \in S \\ (a_i, z, a_j), (a_j, a_{j+1}, a_i) \in S &\Rightarrow (z, a_j, a_{j+1}),\end{aligned}$$

so $(a_j, z, a_{j+1}) \notin S$. The case $j = i+1$ is ruled out by

$$(a_i, z, a_{i+1}), (a_{i+1}, a_{i+2}, a_i) \in S \Rightarrow (z, a_{i+1}, a_{i+2}) \in S$$

and the case $j = i-1$ is similar.

Finally, we put $g(z)$ in $(g(a_n), +\infty)$ if $i = n$, and $(g(a_i), g(a_{i+1}))$ otherwise; an analysis similar to that above shows that $g$ has the desired property.

A-5 (due to Lenny Ng) For $1 \leq n \leq p-1$, $p$ divides $\binom{p}{n}$ and

$$\frac{1}{p}\binom{p}{n} = \frac{1}{n}\frac{p-1}{1}\frac{p-2}{2}\cdots\frac{p-n+1}{n-1} \equiv \frac{(-1)^{n-1}}{n} \pmod{p},$$

where the congruence $x \equiv y \pmod{p}$ means that $x - y$ is a rational number whose numerator, in reduced form, is divisible by $p$. Hence it suffices to show that

$$\sum_{n=1}^{k} \frac{(-1)^{n-1}}{n} \equiv 0 \pmod{p}.$$

We distinguish two cases based on $p \pmod{6}$. First suppose $p = 6r+1$, so that $k = 4r$. Then

$$\begin{aligned}\sum_{n=1}^{4r} \frac{(-1)^{n-1}}{n} &= \sum_{n=1}^{4r} \frac{1}{n} - 2\sum_{n=1}^{2r} \frac{1}{2n} \\ &= \sum_{n=1}^{2r} \left(\frac{1}{n} - \frac{1}{n}\right) + \sum_{n=2r+1}^{3r} \left(\frac{1}{n} + \frac{1}{6r+1-n}\right) \\ &= \sum_{n=2r+1}^{3r} \frac{p}{n(p-n)} \equiv 0 \pmod{p},\end{aligned}$$

since $p = 6r+1$.

Now suppose $p = 6r+5$, so that $k = 4r+3$. A similar argument gives

$$\begin{aligned}
\sum_{n=1}^{4r+3} \frac{(-1)^{n-1}}{n} &= \sum_{n=1}^{4r+3} \frac{1}{n} + 2\sum_{n=1}^{2r+1} \frac{1}{2n} \\
&= \sum_{n=1}^{2r+1} \left(\frac{1}{n} - \frac{1}{n}\right) + \sum_{n=2r+2}^{3r+2} \left(\frac{1}{n} + \frac{1}{6r+5-n}\right) \\
&= \sum_{n=2r+2}^{3r+2} \frac{p}{n(p-n)} \equiv 0 \pmod{p}.
\end{aligned}$$

A-6 We first consider the case $c \leq 1/4$; we shall show in this case $f$ must be constant. The relation

$$f(x) = f(x^2+c) = f((-x)^2+c) = f(-x)$$

proves that $f$ is an even function. Let $r_1 \leq r_2$ be the roots of $x^2+c-x$, both of which are real. Consider the sequence $x_n$ where $x_0$ is arbitrary and $x_{n+1} = x_n^2 + c$. It is a well-known result that $x_n \to r_2$ as long as $x_0 \neq r_1$. To show this, first suppose $x_0 > r_2$; then by induction, $r_2 < x_{n+1} < x_n$ for all $n$, so the $x_n$ tend to a limit $L$, which must be a root of $x^2+c-x$ not less than $r_2$. Hence $L = r_2$. Next, suppose $r_1 < x_0 < r_2$; then by induction, $x_n < x_{n+1} < r_2$ for all $n$, so by the same argument $x_n \to r_2$. Finally, suppose $x < r_1$. As in the first case, as long as $x_n < r_1$, we have $x_{n+1} > x_n$. If all of the $x_n$ were less than $r_1$, they would approach $r_1$ itself. However, if $x_n = r_1 - \epsilon$ for sufficiently small positive $\epsilon$, then $x_{n+1} > r_1$, so we actually are in the second case, and again $x_n \to r_2$.

Since $f(x_0) = f(x_n)$ for all $n$ by the given equation, and $f(x_n) = f(r_2)$ by continuity, we have $f(x) = f(r_2)$ for all $x \neq r_1$, so $f$ is a constant function.

Now suppose $c > 1/4$. Then the sequence $x_n$ defined by $x_0 = 0$ and $x_{n+1} = x_n^2 + c$ is strictly increasing and has no limit point. Thus if we define $f$ on $[x_0, x_1]$ as any continuous function with equal values on the endpoints, and extend the definition from $[x_n, x_{n+1}]$ to $[x_{n+1}, x_{n+2}]$ by the relation $f(x) = f(x^2+c)$, and extend the definition further to $x < 0$ by the relation $f(x) = f(-x)$, the resulting function has the desired property. Moreover, any function clearly has this form.

B-1 Let $[n]$ denote the set $\{1, 2, \ldots, n\}$, and let $f_n$ denote the number of minimal selfish subsets of $[n]$. Then the number of minimal selfish subsets of $[n]$ not containing $n$ is equal to $f_{n-1}$. On the other hand, for any minimal selfish subset of $[n]$ containing $n$, by subtracting 1 from each element, and then taking away the element $n-1$ from the set, we obtain a minimal selfish subset of $[n-2]$ (since 1 and $n$ cannot both occur in a selfish set). Conversely, any minimal selfish subset of $[n-2]$ gives rise to a minimal selfish subset of $[n]$ containing $n$ by the inverse procedure. Hence the number of minimal selfish subsets of $[n]$ containing $n$ is $f_{n-2}$. Thus we obtain $f_n = f_{n-1} + f_{n-2}$. Since $f_1 = f_2 = 1$, we have $f_n = F_n$, where $F_n$ denotes the $n$th term of the Fibonacci sequence.

B-2 By estimating the area under the graph of $\ln x$ using upper and lower rectangles of width 2, we get

$$\int_1^{2n-1} \ln x\,dx \leq 2(\ln(3) + \cdots + \ln(2n-1)) \leq \int_3^{2n+1} \ln x\,dx.$$

Since $\int \ln x\,dx = x\ln x - x + C$, we have, upon exponentiating and taking square roots,

$$\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2}} < (2n-1)^{\frac{2n-1}{2}} e^{-n+1} \leq 1\cdot 3 \cdots (2n-1) \leq (2n+1)^{\frac{2n+1}{2}} \frac{e^{-n+1}}{3^{3/2}} < \left(\frac{2n+1}{e}\right)^{\frac{2n+1}{2}},$$

using the fact that $1 < e < 3$.

B-3 View $x_1, \dots, x_n$ as an arrangement of the numbers $1, 2, \dots, n$ on a circle. We prove that the optimal arrangement is

$$\dots, n-4, n-2, n, n-1, n-3, \dots$$

To show this, note that if $a, b$ is a pair of adjacent numbers and $c, d$ is another pair (read in the same order around the circle) with $a < d$ and $b > c$, then the segment from $b$ to $c$ can be reversed, increasing the sum by

$$ac + bd - ab - cd = (d-a)(b-c) > 0.$$

Now relabel the numbers so they appear in order as follows:

$$\dots, a_{n-4}, a_{n-2}, a_n = n, a_{n-1}, a_{n-3}, \dots$$

where without loss of generality we assume $a_{n-1} > a_{n-2}$. By considering the pairs $a_{n-2}, a_n$ and $a_{n-1}, a_{n-3}$ and using the trivial fact $a_n > a_{n-1}$, we deduce $a_{n-2} > a_{n-3}$. We then compare the pairs $a_{n-4}, a_{n-2}$ and $a_{n-1}, a_{n-3}$, and using that $a_{n-1} > a_{n-2}$, we deduce $a_{n-3} > a_{n-4}$. Continuing in this fashion, we prove that $a_n > a_{n-1} > \ldots > a_1$ and so $a_k = k$ for $k = 1, 2, \dots, n$, i.e. that the optimal arrangement is as claimed. In particular, the maximum value of the sum is

$$\begin{aligned}
&1\cdot 2 + (n-1)\cdot n + 1\cdot 3 + 2\cdot 4 + \cdots + (n-2)\cdot n \\
&\quad= 2 + n^2 - n + (1^2 - 1) + (2^2 - 1) + \cdots + [(n-1)^2 - 1] \\
&\quad= n^2 - n + 2 - (n-1) + \frac{(n-1)n(2n-1)}{6} \\
&\quad= \frac{2n^3 + 3n^2 - 11n + 18}{6}.
\end{aligned}$$

B-4 Suppose such a matrix $A$ exists. If the eigenvalues of $A$ (over the complex numbers) are distinct, then there exists a complex matrix $C$ such that $B = CAC^{-1}$ is diagonal. Consequently, $\sin B$ is diagonal. But then $\sin A = C^{-1}(\sin B)C$ must be diagonalizable, a contradiction. Hence the eigenvalues of $A$ are the same, and $A$ has a conjugate $B = CAC^{-1}$ over the complex numbers of the form

$$\begin{pmatrix} x & y \\ 0 & x \end{pmatrix}.$$

A direct computation shows that

$$\sin B = \begin{pmatrix} \sin x & y \cdot \cos x \\ 0 & \sin x \end{pmatrix}.$$

Since $\sin A$ and $\sin B$ are conjugate, their eigenvalues must be the same, and so we must have $\sin x = 1$. This implies $\cos x = 0$, so that $\sin B$ is the identity matrix, as must be $\sin A$, a contradiction. Thus $A$ cannot exist.

Alternate solution (due to Helfgott and Popa): Define both $\sin A$ and $\cos A$ by the usual power series. Since $A$ commutes with itself, the power series identity

$$\sin^2 A + \cos^2 A = I$$

holds. But if $\sin A$ is the given matrix, then by the above identity, $\cos^2 A$ must equal $\begin{pmatrix} 0 & -2 \cdot 1996 \\ 0 & 0 \end{pmatrix}$ which is a nilpotent matrix. Thus $\cos A$ is also nilpotent. However, the square of any $2 \times 2$ nilpotent matrix must be zero (e.g., by the Cayley-Hamilton theorem). This is a contradiction.

B-5 Consider a $1 \times n$ checkerboard, in which we write an $n$-letter string, one letter per square. If the string is balanced, we can cover each pair of adjacent squares containing the same letter with a $1 \times 2$ domino, and these will not overlap (because no three in a row can be the same). Moreover, any domino is separated from the next by an even number of squares, since they must cover opposite letters, and the sequence must alternate in between.

Conversely, any arrangement of dominoes where adjacent dominoes are separated by an even number of squares corresponds to a unique balanced string, once we choose whether the string starts with $X$ or $O$. In other words, the number of balanced strings is twice the number of acceptable domino arrangements.

We count these arrangements by numbering the squares $0, 1, \ldots, n-1$ and distinguishing whether the dominoes start on even or odd numbers. Once this is decided, one simply chooses whether or not to put a domino in each eligible position. Thus we have $2^{\lfloor n/2 \rfloor}$ arrangements in the first case and $2^{\lfloor (n-1)/2 \rfloor}$ in the second, but note that the case of no dominoes has been counted twice. Hence the number of balanced strings is

$$2^{\lfloor (n+2)/2 \rfloor} + 2^{\lfloor (n+1)/2 \rfloor} - 2.$$

B-6 We will prove the claim using only that the convex hull of the points $(a_i, b_i)$ contains the origin. The first and second members of the sum on the left, divided by $x$ and $y$, respectively, are the partial derivatives of

$$f(x, y) = x^{a_1} y^{b_1} + x^{a_2} y^{b_2} + \cdots + x^{a_n} y^{b_n},$$

and so it suffices to show $f$ has a critical point. We will in fact show $f$ has a global minimum.

Let $u = \log x, v = \log y$, so that $f(x,y) = \sum_i \exp(a_i u + b_i v)$. We trivially have

$$f(x,y) \geq \exp\left(\max_i (a_i u + b_i v)\right).$$

Note that this maximum is positive for $(u,v) \neq (0,0)$: if we had $a_i u + b_i v < 0$ for all $i$, then the subset $ur + vs < 0$ of the $rs$-plane would be a half-plane containing all of the points $(a_i, b_i)$, whose convex hull would then not contain the origin, a contradiction.

The function $\max_i (a_i u + b_i v)$ is clearly continuous on the unit circle $u^2 + v^2 + 1$, which is compact. Hence it has a global minimum $M > 0$, and so for all $u, v$,

$$\max_i (a_i u + b_i v) \geq M\sqrt{u^2 + v^2}.$$

In particular, $f \geq n + 1$ on the disk of radius $\sqrt{(n+1)/M}$. Since $f = n$ for $u = v = 0$, the infimum of $f$ is the same over the entire $uv$-plane as over this disk, which again is compact. Hence $f$ attains its infimal value at some point in the disk, which is the desired global minimum.