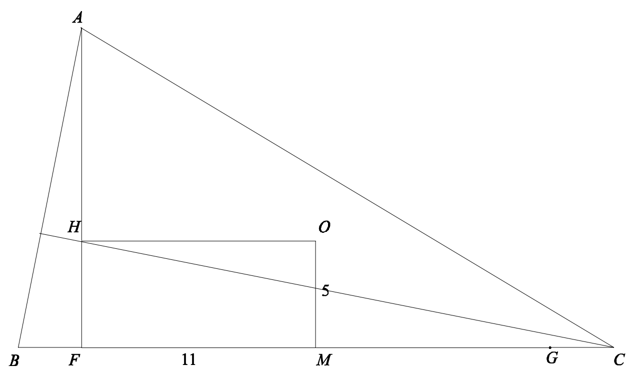
Putnam 1997 (Problems and Solutions)

A1. A rectangle, $HOMF$, has sides $HO = 11$ and $OM = 5$. A triangle ABC has H as the intersection of the altitudes, O the center of the circumscribed circle, M the midpoint of BC, and F the foot of the altitude from A. What is the length of BC?

Solution. In the figure below, let G be such that $MG = 11$, let $x = BF = GC$, and let $y = AH$.



Since O is the circumcenter, we have $y^2 + 11^2 = 5^2 + (11 + x)^2$. The slope of the line AB is $(5 + y)/x$ and the slope of HC is $-5/(22 + x)$. Since these lines meet at right angles, the product of the slopes is -1. Thus, $\frac{5+y}{x}\frac{5}{22+x} = 1$ or $25 + 5y = 22x + x^2$. Hence

$$y^2 + 11^2 = 5^2 + (11 + x)^2 = 25 + 11^2 + 22x + x^2 = 25 + 11^2 + 25 + 5y,$$

or $y^2 - 5y - 50 = 0$ or $(y - 10)(y + 5) = 0$. Thus, $y = 10$, $x = 3$ and $BC = 22 + 6 = 28$.

A2. Players $1, 2, 3, ..., n$ are seated around a table and each has a single penny. Player 1 passes a penny to Player 2, who then passes two pennies to Player 3. Player 3 then passes one penny to player 4, who passes two pennies to Player 5, and so on, players alternately passing one penny or two to the next player who still has pennies. A player who runs out of pennies drops out of the game and leaves the table. Find an infinite set of numbers n for which some player ends up with all n pennies.

Solution. Suppose that at some point there are an odd number greater than 1 of remaining players, say $p_1, p_2, ..., p_{2k+1}$ $(k \geq 1)$, none in danger of dropping out on their next pass. Then there is an odd number of passes starting with p_1's pass to p_2 and ending with p_{2k+1}'s pass to p_1. Thus, if p_1 passes 2 coins he will receive 2 coins, and if he passes 1 he will receive 1. The same is true for any other player and the game will not terminate. Thus, we must avoid this situation, or we are "stuck" with a nonterminating game. Suppose that to begin with there are $n = 2m$ (where for notational convenience, $m \geq 3$) players, say $p_1, ..., p_{2m}$. Note that p_1 drops out and p_2 passes his two and drops out, p_3 passes 1 and ends up with 2, p_4 drops out, but p_5 ends up with 2. Continuing, $p_3, p_5, ..., p_{2m-1}$ each have 2 coins and p_{2m} drops out when he passes 2 to p_3 who now has 4. There are now $m-1$ players and we are stuck if $m-1$ is odd. Thus, we need $m - 1 = 2m_1$ for some m_1. Relabel the players $q_1, ..., q_{2m_1}$. Now q_1 has 4 coins, the rest have 2, and q_1 starts this "stage" by passing 1 to q_2. Since there are an even number $(2m_1)$ of players, q_1 will eventually receive 2 (for a total of 5) and pass 1 again to q_2 to begin the second "round" of the stage, but then q_2 drops out, as do $q_4, q_6, ..., q_{2m_1}$. When q_{2m_1} passes 2 to q_1 and drops out, q_1 will have 7 and the rest $q_3, q_5, ..., q_{2m_1-1}$ each have 4. At the beginning of this new stage, there are now m_1 players left which must be even, or else $m_1 = 1$ (i.e., we have a winner) or we are stuck. Continuing, we see that the game terminates if and only if m_1 is a power of 2, since half of the players (those repeatedly passing 2) eventually drop out at each stage which necessarily must have an even number of players if not a winner. Observe that all the players passing 2 in a stage, have the same number of coins, and so they all drop out in the same final round of the stage. Thus, a game with $n = 2m$ players terminates if and only if $m - 1 = 2m_1 = 2^k$ for some $k \geq 0$, or $n = 2m = 2\left(2^k + 1\right) = 2^h + 2$, $h = 2, 3, 4, ...$ will yield terminating games. Of course, $n = 2$ and $n = 4$ games also terminate. Incidentally, for $n = 2^h + 2$, the original player p_3 is the winner since each stage ends with p_3 receiving 2.

A3. Evaluate

$$\int_0^\infty \left(x - \frac{x^3}{2} + \frac{x^5}{2\cdot 4} - \frac{x^7}{2\cdot 4\cdot 6} + \cdots\right)\left(1 - \frac{x^2}{2^2} + \frac{x^4}{2^2\cdot 4^2} + \frac{x^6}{2^2\cdot 4^2\cdot 6^2} + \cdots\right) dx$$

Solution. The first series is $xe^{-\frac{1}{2}x^2}$. For the second series, we use the fact that for $n = 0, 1, 2, ...$,

$$\begin{aligned} \int_0^\pi \cos^{2n}\theta\, d\theta &= \frac{1\cdot 3\cdot 5\cdots(n-1)}{2\cdot 4\cdot 6\cdots n}\pi = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdots(2n-1)\,2n}{2^2\cdot 4^2\cdot 6^2\cdots(2n)^2}\pi \\ &= \frac{(2n)!}{2^2\cdot 4^2\cdot 6^2\cdots(2n)^2}\pi. \end{aligned}$$

Then second series is (using $\int_0^\pi \cos^{2n+1}\theta\, d\theta = 0$),

$$\begin{aligned} \sum_{n=0}^\infty \frac{\frac{1}{\pi}\int_0^\pi \cos^{2n}\theta\, d\theta}{(2n)!} x^{2n} &= \sum_{n=0}^\infty \frac{\frac{1}{\pi}\int_0^\pi \cos^k\theta}{k!}\, d\theta\, x^k = \frac{1}{\pi}\int_0^\pi \sum_{n=0}^\infty \frac{(x\cos\theta)^k}{k!}\, d\theta \\ &= \frac{1}{\pi}\int_0^\pi e^{x\cos\theta}\, d\theta. \end{aligned}$$

Thus, the desired integral is

$$\begin{aligned} &\int_0^\infty xe^{-\frac{1}{2}x^2}\frac{1}{\pi}\int_0^\pi e^{x\cos\theta}\, d\theta\, dx = \frac{1}{\pi}\int_0^\pi\int_0^\infty e^{-\frac{1}{2}r^2}e^{r\cos\theta}\, r\, dr d\theta \\ =\; &\frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty e^{-\frac{1}{2}\left(x^2+y^2\right)}e^x\, dxdy = \frac{1}{\pi}e^{\frac{1}{2}}\int_{-\infty}^\infty e^{-\frac{1}{2}\left(x^2-2x+1\right)}\, dx\int_0^\infty e^{-\frac{1}{2}y^2}dy \\ =\; &\frac{1}{\pi}e^{\frac{1}{2}}\int_{-\infty}^\infty e^{-\frac{1}{2}(x-1)^2}\, dx\int_0^\infty e^{-\frac{1}{2}y^2}dy = \frac{1}{\pi}e^{\frac{1}{2}}\sqrt{2\pi}\left(\tfrac{1}{2}\sqrt{2\pi}\right) = \sqrt{e}. \end{aligned}$$

A4. Let G be a group with identity e and $\phi : G \to G$ be a function such that

$$\phi(g_1)\phi(g_2)\phi(g_3) = \phi(h_1)\phi(h_2)\phi(h_3)$$

whenever $g_1 g_2 g_3 = e = h_1 h_2 h_3$. Prove that there exists an element a in G such that $\psi(x) = a\phi(x)$ is a homomorphism (that is, $\psi(xy) = \psi(x)\psi(y)$ for all x and y in G).

Solution. Since $x^{-1}xe = e^3$, we have

$$\phi\left(x^{-1}\right)\phi(x)\phi(e) = \phi(e)^3 \quad \text{or} \quad \phi\left(x^{-1}\right)\phi(x) = \phi(e)^2 \quad \text{or} \quad \phi\left(x^{-1}\right)^{-1} = \phi(x)\phi(e)^{-2}.$$

Since $yy^{-1}e = e^3$, we have

$$\phi(y)\phi\left(y^{-1}\right)\phi(e) = \phi(e)^3 \quad \text{or} \quad \phi(y)\phi\left(y^{-1}\right) = \phi(e)^2 \quad \text{or} \quad \phi\left(y^{-1}\right)^{-1} = \phi(e)^{-2}\phi(y).$$

Since $x^{-1}(xy)y^{-1} = eee$,we have

$$\phi\left(x^{-1}\right)\phi(xy)\phi\left(y^{-1}\right) = \phi(e)^3,$$

and so

$$\begin{aligned} \phi(xy) &= \phi\left(x^{-1}\right)^{-1}\phi(e)^3\phi\left(y^{-1}\right)^{-1} = \phi(x)\phi(e)^{-2}\phi(e)^3\phi(e)^{-2}\phi(y) \\ &= \phi(x)\phi(e)^{-1}\phi(y). \end{aligned}$$

A5. Let N_n denote the number of ordered n-tuples of positive integers $(a_1, a_2, \ldots, a_n)$ such that $1/a_1 + 1/a_2 + \ldots + 1/a_n = 1$. Determine whether N_{10} is even or odd.

Solution. Consider the involution of the set of solutions

$$(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}) \mapsto (a_2, a_1, a_4, a_3, a_6, a_5, a_8, a_7, a_{10}, a_9)$$

The number of points that are not fixed is even. Thus, we need only to consider the solutions of the form $(a_1, a_1, a_3, a_3, a_5, a_5, a_7, a_7, a_9, a_9)$. On this set of remaining solutions, consider the involution

$$(a_1, a_1, a_3, a_3, a_5, a_5, a_7, a_7, a_9, a_9) \mapsto (a_3, a_3, a_1, a_1, a_7, a_7, a_5, a_5, a_9, a_9)$$

The points that are fixed are of the form

$$(a_1, a_1, a_1, a_1, a_5, a_5, a_5, a_5, a_9, a_9).$$

By one more involution we need only consider the number of solutions of the form

$$(a_1, a_1, a_1, a_1, a_1, a_1, a_1, a_1, a_9, a_9).$$

For these $\frac{8}{a_1} + \frac{2}{a_9} = 1$, and so $a_9 = \frac{2a_1}{a_1 - 8}$. The set of possible pairs (a_1, a_9) is $\{(9, 18), (10, 10), (12, 6), (16, 4), (24, 3)\}$. Thus, N_{10} is odd.

A6. For a positive integer n and any real number c, define x_k recursively by $x_0 = 0$, $x_1 = 1$, and for $k \geq 0$,

$$x_{k+2} = \frac{cx_{k+1} - (n-k)x_k}{k+1}.$$

Fix n and then take c to be the largest value for which $x_{n+1} = 0$. Find x_k in terms of n and k, $1 \leq k \leq n$.

Solution. The recurrence relation is as one might find while solving a differential equation using power series. We have

$$\begin{aligned}
(k+1)\,x_{k+2} &= cx_{k+1} - (n-k)x_k \\
\sum_{k=0}^{\infty} x_{k+2}\,(k+1)\,t^k &= c\sum_{k=0}^{\infty} x_{k+1}t^k - n\sum_{k=0}^{\infty} x_k t^k + \sum_{k=0}^{\infty} k x_k t^k \\
\sum_{k=0}^{\infty} x_{k+2}\,(k+1)\,t^k &= c\sum_{k=0}^{\infty} x_{k+1}t^k - (n-1)\sum_{k=0}^{\infty} x_k t^k + t^2\sum_{k=1}^{\infty} (k-1)\,x_k t^{k-2}
\end{aligned}$$

Let $f(t) = \sum_{k=0}^{\infty} x_{k+1}t^k = 1 + \sum_{k=0}^{\infty} x_{k+2}t^{k+1} = \sum_{k=1}^{\infty} x_k t^{k-1}$. Then the above says

$$\begin{aligned}
f'(t) &= cf(t) - (n-1)\,tf(t) + t^2 f'(t) \\
\frac{f'(t)}{f(t)} &= \frac{c-(n-1)t}{1-t^2} = \frac{B}{1+t} + \frac{A}{1-t} \\
&= \frac{\frac{1}{2}(n-1+c)}{1+t} - \frac{\frac{1}{2}(n-1-c)}{1-t}.
\end{aligned}$$

Hence

$$f(t) = (1-t)^{\frac{1}{2}(n-1-c)}\,(1+t)^{\frac{1}{2}(n-1+c)}.$$

Now $f(t)$ will be a polynomial of degree $n-1$ if $\frac{1}{2}(n-1+c)$ and $\frac{1}{2}(n-1-c)$ are nonnegative integers, say j and $n-1-j$ (i.e., $j = 0, ..., n-1$). Since $f(t) = \sum_{k=0}^{\infty} x_{k+1}t^k$, we would then have $x_{n+1} = 0$. Since $j = \frac{1}{2}(n-1+c)$, $c = 2j - n + 1$. Thus, some possible values for c for which $x_{n+1}(n,c) = 0$ are the n values

$$-n+1, -n+3, ..., 2(n-1) - n + 1 = n-1.$$

However, it is clear that $x_k(n,c)$ is a polynomial of degree at most $k-1$ in c. Indeed, $x_2(n,c) = c$, and from $x_{k+2} = \frac{cx_{k+1}-(n-k)x_k}{k+1}$ the result follows from induction. Thus, the values the above values are the only values of c for which $x_{n+1}(n,c) = 0$. The largest of these values is $c = n-1$. Then

$$f(t) = (1-t)^{\frac{1}{2}(n-1-c)}\,(1+t)^{\frac{1}{2}(n-1+c)} = (1+t)^{n-1},$$

and $f(t) = \sum_{k=0}^{\infty} x_{k+1} t^k \Rightarrow x_{k+1} = \binom{n-1}{k}$ or

$$x_k = \begin{cases} 0 & k = 0 \\ 1 & k = 1 \\ \binom{n-1}{k-1} & k = 2, ..., n \\ 0 & k > n \end{cases} .$$

B1. Let $\{x\}$ denote the distance between the real number x and the nearest integer. For each positive integer n, evaluate

$$F_n = \sum_{m=1}^{6n-1} \min(\{\frac{m}{6n}\}, \{\frac{m}{3n}\}).$$

(Here $\min(a, b)$ denotes the minimum of a and b.)

Solution. We have

$$\{\frac{m}{6n}\} = \begin{cases} \frac{m}{6n} & 1 \le m \le 3n \\ 1 - \frac{m}{6n} & 3n \le m \le 6n-1 \end{cases}$$

$$\{\frac{m}{3n}\} = \begin{cases} \frac{m}{3n} & 1 \le m \le 3n/2 \\ 1 - \frac{m}{3n} & 3n/2 \le m \le 3n \\ \frac{m}{3n} - 1 & 3n \le m \le 3n\frac{3}{2} = \frac{9n}{2} \\ 2 - \frac{m}{3n} & \frac{9n}{2} \le m \le 6n-1 \end{cases}.$$

Now

$$\min(\{\frac{m}{6n}\}, \{\frac{m}{3n}\}) = \begin{cases} \frac{m}{6n} & 1 \le m \le 3n/2 \\ \min\left(\frac{m}{6n}, 1 - \frac{m}{3n}\right) & 3n/2 \le m \le 3n \\ \min\left(1 - \frac{m}{6n}, \frac{m}{3n} - 1\right) & 3n \le m \le 3n\frac{3}{2} = \frac{9n}{2} \\ \min\left(1 - \frac{m}{6n}, 2 - \frac{m}{3n}\right) & \frac{9n}{2} \le m \le 6n-1 \end{cases}$$

Using

$$\begin{aligned} \frac{m}{6n} &\le 1 - \frac{m}{3n} \Leftrightarrow m \le 2n, \\ 1 - \frac{m}{6n} &\le \frac{m}{3n} - 1 \Leftrightarrow m \ge 4n \\ 1 - \frac{m}{6n} &\le 2 - \frac{m}{3n} \Leftrightarrow m \le 6n, \end{aligned}$$

we have

$$\min(\{\frac{m}{6n}\}, \{\frac{m}{3n}\}) = \begin{cases} \frac{m}{6n} & 1 \le m \le 3n/2 \\ \frac{m}{6n} & 3n/2 \le m \le 2n \\ 1 - \frac{m}{3n} & 2n \le m \le 3n \\ \frac{m}{3n} - 1 & 3n \le m \le 4n \\ 1 - \frac{m}{6n} & 4n \le m \le \frac{9}{2}n \\ 1 - \frac{m}{6n} & \frac{9}{2}n \le m \le 6n-1 \end{cases} = \begin{cases} \frac{m}{6n} & 1 \le m \le 2n-1 \\ 1 - \frac{m}{3n} & 2n \le m \le 3n-1 \\ \frac{m}{3n} - 1 & 3n \le m \le 4n-1 \\ 1 - \frac{m}{6n} & 4n \le m \le 6n-1 \end{cases}$$

Hence,

$$
\begin{aligned}
F_n &= \sum_{m=1}^{6n-1} \min(\{\frac{m}{6n}\},\{\frac{m}{3n}\}) \\
&= \sum_{m=1}^{2n-1} \frac{m}{6n} + \sum_{m=2n}^{3n-1} \left(1-\frac{m}{3n}\right) + \sum_{m=3n}^{4n-1} \left(\frac{m}{3n}-1\right) + \sum_{m=4n}^{6n-1} \left(1-\frac{m}{6n}\right) \\
&= \frac{1}{6n}\left(\sum_{m=1}^{2n-1} m - 2\sum_{m=2n}^{3n-1} m + 2\sum_{m=3n}^{4n-1} m - \sum_{m=4n}^{6n-1} m\right) \\
&\quad + (3n-1-2n+1) - (4n-1-3n+1) + (6n-1-4n+1) \\
&= \frac{1}{6n}\left(\tfrac{1}{2}(2n)(2n-1) - (5n-1)\,n + (7n-1)\,n - \tfrac{1}{2}(10n-1)\,2n\right) + 2n \\
&= n.
\end{aligned}
$$

B2. Let f be a twice-differentiable real-valued function satisfying

$$f(x) + f''(x) = -xg(x)f'(x),$$

where $g(x) \geq 0$ for all real x. Prove that $|f(x)|$ is bounded.

Solution. Multiplying by $f'(x)$, we get

$$\tfrac{1}{2}\frac{d}{dx}\left(f(x)^2 + f'(x)^2\right) = f(x)f'(x) + f'(x)f''(x) = -xg(x)f'(x)^2.$$

Thus, $f(x)^2 + f'(x)^2$ is nondecreasing for $x \leq 0$ and nonincreasing for $x \geq 0$. Hence

$$f(x)^2 \leq f(x)^2 + f'(x)^2 \leq f(0)^2 + f'(0)^2.$$

B3. For each positive integer n, write the sum $\sum_{m=1}^{n} 1/m$ in the form p_n/q_n, where p_n and q_n are relatively prime positive integers. Determine all n such that 5 does not divide q_n.

Solution. We may write the sum $S_n := \sum_{m=1}^{n} 1/m$ in the form

$$\begin{aligned} S_n &= \sum_{5\nmid m}^{n} 1/m + \frac{1}{5}\sum_{5\nmid m}^{\lfloor n/5\rfloor} 1/m + \frac{1}{5^2}\sum_{5\nmid m}^{\lfloor n/5^2\rfloor} 1/m + ... \\ &= \sum_{k=0}^{K} \frac{1}{5^k}\left(\sum_{5\nmid m}^{\lfloor n/5^k\rfloor} 1/m\right) = \sum_{k=0}^{K} \frac{1}{5^k} F_{\lfloor n/5^k\rfloor}, \end{aligned}$$

where 5^K is the largest power of 5 not greater than n and

$$F_M := \sum_{5\nmid m}^{M} \frac{1}{m}.$$

Consider a sum of reciprocals of integers $n_1, ..., n_k$ not divisible by 5, say

$$\frac{1}{n_1} + \frac{1}{n_2} + ... + \frac{1}{n_k} = \frac{n_2...n_k + n_1 n_3...n_k + ... + n_1 n_2 ... n_{k-1}}{n_1 n_2 ... n_k} = \frac{N}{D}$$

with $D = n_1 n_2 ... n_k$. Let $\overline{n}_i$ be the associate of n_i; i.e., $\overline{n}_i n_i \equiv 1 \bmod (5)$. Note that

$$\begin{aligned} D(\overline{n}_1 + \overline{n}_2 + ... + \overline{n}_k) &= n_1 n_2 ... n_k (\overline{n}_1 + \overline{n}_2 + ... + \overline{n}_k) \\ &\equiv n_2...n_k + n_1 n_3 ... n_k + ... + n_1 n_2 ... n_{k-1} = N \pmod 5 \end{aligned}$$

Thus,

$$D(\overline{n}_1 + \overline{n}_2 + ... + \overline{n}_k) \equiv N \pmod 5$$

Since $5 \nmid D$ we have

$$5|N \Leftrightarrow \overline{n}_1 + \overline{n}_2 + ... + \overline{n}_k \equiv 0 \pmod 5$$

Note that

$$\overline{1} = 1, \overline{2} = 3, \overline{3} = 2, \overline{4} = 4$$

Working mod 5, we have

$$\begin{aligned} \overline{1} &= 1, \\ \overline{1} + \overline{2} &= -1, \\ \overline{1} + \overline{2} + \overline{3} &= 1, \\ \overline{1} + \overline{2} + \overline{3} + \overline{4} &= 0 \\ \overline{1} + \overline{2} + \overline{3} + \overline{4} + \overline{6} &= 1 \\ \overline{1} + \overline{2} + \overline{3} + \overline{4} + \overline{6} + \overline{7} &= -1 \\ \overline{1} + \overline{2} + \overline{3} + \overline{4} + \overline{6} + \overline{7} + \overline{8} &= 1 \\ \overline{1} + \overline{2} + \overline{3} + \overline{4} + \overline{6} + \overline{7} + \overline{8} + \overline{9} &= 0, \text{ etc..} \end{aligned}$$

Writing $F_M = \frac{N_M}{D_M}$, we have $N_M \equiv 1, -1, 1, 0, 0 \pmod 5$ if $M \equiv 1, 2, 3, 4, 5 \pmod 5$, respectively. Note that if $M \equiv 5$, then $F_M = F_{M-1}$ and $N_M = N_{M-1} \equiv 0 \pmod 5$. Recall

$$S_n = \sum_{k=0}^{K} \frac{1}{5^k} F_{\lfloor n/5^k \rfloor} \qquad (*)$$

Consider the last term

$$\frac{1}{5^K} F_{\lfloor n/5^K \rfloor}$$

in $(*)$. We have $\lfloor n/5^K \rfloor = 1, 2, 3$ or 4. If $\lfloor n/5^K \rfloor = 1, 2$,or 3, then the numerator of $F_{\lfloor n/5^K \rfloor}$ is not divisible by 5. In this case, S_n must have a factor of at least 5^K in its denominator. Thus, $K = 0$ in this case and $n = 1, 2$,or 3.

Henceforth, suppose that $\lfloor n/5^K \rfloor = 4$. In this case, we have $F_4 = 1 + 1/2 + 1/3 + 1/4 = \frac{25}{12}$ and $\frac{1}{5^K} F_{\lfloor n/5^K \rfloor} = \frac{1}{12 \cdot 5^{K-2}}$. If $K = 0$, then $S_4 = F_4 = \frac{25}{12}$ and $n = 4$ has the desired property. If $K \geq 1$, the preceding term in $(*)$ is $\frac{1}{5^{K-1}} F_{\lfloor n/5^{K-1} \rfloor}$, and

$$\lfloor n/5^K \rfloor = 4 \Rightarrow n/5^K = 4, \frac{21}{5}, \frac{22}{5}, \frac{23}{5}, \frac{24}{5} \Rightarrow \lfloor n/5^{K-1} \rfloor = 20, 21, ..., 24.$$

If $\lfloor n/5^{K-1} \rfloor = 21, 22, 23$, then $\frac{1}{5^{K-1}} F_{\lfloor n/5^{K-1} \rfloor}$ has numerator not divisible by 5 and the denominator has a factor of 5^{K-1}. As all other denominators in the sum have lower powers of 5 (in particular the last term is $\frac{1}{12 \cdot 5^{K-2}}$), in this case S_n will have denominator divisible by 5 unless $K \leq 1$. Thus, in the case $\lfloor n/5^{K-1} \rfloor = 21, 22, 23$, we have $n = \lfloor n \rfloor = \lfloor n/5^{K-1} \rfloor = 21, 22, 23$. Now suppose that $\lfloor n/5^{K-1} \rfloor = 20, 24$. We have $F_{20} = \frac{16456225}{5173168}$ and $F_{24} = \frac{399698125}{118982864}$. As the numerators of F_{20} and F_{24} are divisible by 25, the term $\frac{1}{5^{K-1}} F_{\lfloor n/5^{K-1} \rfloor}$ has denominator with a power of 5 of at least $K - 3$. The sum of the two terms in $(*)$ before and after this term is

$$\frac{1}{5^{K-2}} F_{\lfloor n/5^{K-2} \rfloor} + \frac{1}{5^K} F_{\lfloor n/5^K \rfloor} = \frac{1}{5^{K-2}} \left(F_{\lfloor n/5^{K-2} \rfloor} + \frac{1}{12} \right)$$

Since $\overline{12} \equiv 3$ and the numerator of $F_{\lfloor n/5^{K-2} \rfloor}$ can be only be congruent to ± 1 or 0, the denominator of this sum has a factor of at least 5^{K-2}, and all of the other terms have denominators with lower powers of 5. Thus, we must have $K = 1$ or 2. For $K = 1$, $n = 20, 24$ and for $K = 2$,we have $\lfloor n/5 \rfloor = 20, 24 \Rightarrow n = 100 - 104, 120 - 124$. Thus, in view of all of the above, the only values for n with the desired property are $1 - 4, 20 - 24, 100 - 104$, and $120 - 124$.

B4. Let $a_{m,n}$ denote the coefficient of x^n in the expansion of $(1+x+x^2)^m$. Prove that for all $k \geq 0$,

$$0 \leq \sum_{i=0}^{\lfloor \frac{2k}{3} \rfloor} (-1)^i a_{k-i,i} \leq 1.$$

Solution. Note that since the degree of $(1+x+x^2)^m$ is $2m$, we have $a_{m,n} = 0$ for $n > 2m$. Hence $a_{k-i,i} = 0$ for $i > 2(k-i)$ or $3i > 2k$. Hence the upper limit on the sum only serves to eliminate terms which would be 0 or undefined. Let us define $a_{m,n} = 0$ for $n < 0$ or $m < 0$. Thus in the sum above and all sums below, we may take the index to run over all integers.

$$\begin{aligned}
\sum_n a_{m+1,n} x^n &= (1+x+x^2)^{m+1} = (1+x+x^2)(1+x+x^2)^m \\
&= (1+x+x^2) \sum_k a_{m,k} x^k = \sum_k a_{m,k} (x^k + x^{k+1} + x^{k+2}) \\
&= \sum_n a_{m,n} x^n + \sum_n a_{m,n-1} x^n + \sum_n a_{m,n-2} x^n \\
&= \sum_n (a_{m,n} + a_{m,n-1} + a_{m,n-2}) x^n.
\end{aligned}$$

Thus,

$$a_{m+1,n} = a_{m,n} + a_{m,n-1} + a_{m,n-2}.$$

Let $s_k := \sum_{i=0}^{\lfloor \frac{2k}{3} \rfloor} (-1)^i a_{k-i,i} = \sum_i (-1)^i a_{k-i,i}$. Then

$$\begin{aligned}
s_{k+1} \quad : &= \sum_i (-1)^i a_{k+1-i,i} = \sum_i (-1)^i \left(a_{k-i,i} + a_{k-i,i-1} + a_{k-i,i-2} \right) \\
&= \sum_i \left((-1)^i a_{k-i,i} - (-1)^{i-1} a_{(k-1)-(i-1),i-1} + (-1)^{i-2} a_{(k-2)-(i-2),i-2} \right) \\
&= s_k - s_{k-1} + s_{k-2}
\end{aligned}$$

We have

$$s_{-2} = \sum_i (-1)^i a_{-2-i,i} = 0, \; s_{-1} = \sum_i (-1)^i a_{-1-i,i} = 0, \; s_0 = \sum_i (-1)^i a_{0-i,i} = a_{0,0} = 1$$

Thus,

$$\begin{aligned}
s_1 &= s_0 - s_{-1} + s_{-2} = 1 \\
s_2 &= s_1 - s_0 + s_{-1} = 0 \\
s_3 &= s_2 - s_1 + s_0 = 0 \\
s_4 &= s_3 - s_2 + s_1 = 1 \\
s_5 &= s_4 - s_3 + s_2 = 1
\end{aligned}$$

Assume that $s_{4k} = 1, s_{4k+1} = 1, s_{4k+2} = 0, s_{4k+3} = 0$. Then

$$\begin{aligned}
s_{4(k+1)} &= s_{4k+4} = s_{4k+3} - s_{4k+2} + s_{4k+1} = 0 - 0 + 1 = 1, \\
s_{4(k+1)+1} &= s_{4k+5} = s_{4k+4} - s_{4k+3} + s_{4k+2} = 1 - 0 + 0 = 1, \\
s_{4(k+1)+2} &= s_{4k+6} = s_{4k+5} - s_{4k+4} + s_{4k+3} = 1 - 1 + 0 = 0, \text{ and} \\
s_{4(k+1)+3} &= s_{4k+7} = s_{4k+6} - s_{4k+5} + s_{4k+4} = 0 - 1 + 1 = 0.
\end{aligned}$$

Thus, s_k is 0 or 1 for all $k \geq 0$, and in particular $0 \leq s_k \leq 1$.

B5. Prove that for $n \geq 2$,

$$2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\Big\}\, n \equiv 2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\Big\}\, n-1 \pmod{n}$$

Solution. Let $t_1 := 2$ and let $t_n := 2^{t_{n-1}}$. We are to show $t_n = 2^{t_{n-1}} \equiv t_{n-1} \pmod{n}$. We first check the result in the case $n = 2^k$. Note that

$$t_{n-2} \geq 2^k \Rightarrow 2^{t_{n-1}} \equiv 2^{t_{n-2}} \left(\bmod 2^k\right) \Longleftrightarrow t_n \equiv t_{n-1} \left(\bmod 2^k\right)$$

Thus, for the case $n = 2^k$, it suffices to check that $t_{2^k-2} \geq 2^k$. This is true if $k = 1$. If it is true for $k \geq 1$, then

$$t_{2^{k+1}-2} > t_{2^k-1} = 2^{t_{2^k-2}} \geq 2^{2^k} \geq 2^{k+1},$$

and so $t_{2^k-2} \geq 2^k$ by induction. We handle the case $n = 2^k d$ where d is odd, by using induction on d. The case $d = 1$ has been done. First note that since $(d, 2) = 1$,

$$2^{\varphi(d)} \equiv 1 \pmod{d}.$$

Thus, we have the key observation that

$$\begin{aligned} t_{m-2} &\equiv t_{m-1} \left(\bmod \varphi(d)\right) \Rightarrow 2^{t_{m-1}-t_{m-2}} \equiv 1 \pmod{d} \\ &\Rightarrow 2^{t_{m-2}} \equiv 2^{t_{m-1}} \pmod{d} \Rightarrow t_{m-1} \equiv t_m \pmod{d}. \end{aligned}$$

Now

$$t_{n-1} \equiv t_n \left(\bmod 2^k d\right) \Leftrightarrow \begin{cases} t_{n-1} \equiv t_n \left(\bmod 2^k\right) \\ \text{and} \\ t_{n-1} \equiv t_n \pmod{d} \end{cases}$$

Since $n \geq 2^k$, we have already shown $t_{n-1} \equiv t_n \left(\bmod 2^k\right)$. Thus, we need only show that $t_{n-1} \equiv t_n \pmod{d}$ and by the key observation it suffices to show that $t_{n-2} \equiv t_{n-1} \left(\bmod \varphi(d)\right)$. We have

$$d - 1 \geq n' := \varphi(d) = 2^{k'} d',$$

where d' is odd. Since $d' < d$, by induction we have

$$t_{n'-1} \equiv t_{n'} \pmod{n'} \quad \text{or} \quad t_{n'-1} \equiv t_{n'} \left(\bmod \varphi(d)\right).$$

However, we need $t_{n-2} \equiv t_{n-1} \left(\bmod \varphi(d)\right)$. Since $n - 1 \geq d - 1 \geq n'$, we have $n - 2 \geq n' - 1$. Hence, if we could show the stronger result that not only $t_{n-1} \equiv t_n \pmod{n}$ but also

$$t_{n-1} \equiv t_n \equiv t_{n+1} \equiv t_{n+2} \equiv \cdots \pmod{n},$$

(ad infinitum), then we would have

$$t_{n'-1} \equiv t_{n'} \equiv \cdots \equiv t_{n-1} \equiv t_n \equiv t_{n+1} \equiv \cdots \pmod{\varphi(d)}$$

The stronger result $t_{n-1} \equiv t_n \equiv t_{n+1} \equiv t_{n+2} \equiv \cdots \pmod{n}$ is true in the base case $n = 2^k$. Then assuming it holds for $d' < d$ inductively, we have

$$t_{n'-1} \equiv t_{n'} \equiv \cdots \equiv t_{n-1} \equiv t_n \equiv t_{n+1} \equiv \cdots \pmod{\varphi(d)}$$

in which case by the key observation,

$$t_{n'} \equiv \cdots \equiv t_{n-1} \equiv t_n \equiv t_{n+1} \equiv \cdots \pmod{d}.$$

We already know that

$$t_n \equiv t_{n+1} \equiv \cdots \left(\bmod 2^k\right).$$

Thus,

$$t_n \equiv t_{n+1} \equiv t_{n+2} \equiv \cdots \left(\bmod 2^k d\right),$$

and we have proven this stronger result by induction.

B6. The dissection of the 3–4–5 triangle shown below (into four congruent right triangles similar to the original) has diameter 5/2. Find the least diameter of a dissection of this triangle into four parts. (The diameter of a dissection is the least upper bound of the distances between pairs of points belonging to the same part.)

Solution. Lower bounds on the least diameter of a dissection of the triangle into 4 parts can be obtained by selecting 5 points and taking the smallest distance between pairs. Indeed, any dissection into four parts must have one region containing 2 of the 5 points and that region has diameter at least the distance between the two points. Thus, optimistically, we seek 5 points so that the minimum distance between pairs is as large as possible. Then we attempt to find a dissection where no region has diameter larger than this minimum distance.

Let A$= (0,4)$,B$= (0,0)$,C$= (3,0)$. The point on the segment AC which is distance t from $(3,0)$ is $\frac{1}{5}t\,(0,4) + \left(1-\frac{1}{5}t\right)(3,0) = \left(3-\frac{3}{5}t, \frac{4}{5}t\right)$. The square of the distance from this point to the point $(0, 4-t)$ on AB (at distance t from A) is

$$\left\| \left(3-\frac{3}{5}t, \frac{4}{5}t\right) - (0, 4-t) \right\|^2 = 25 - 18t + \frac{18}{5}t^2$$

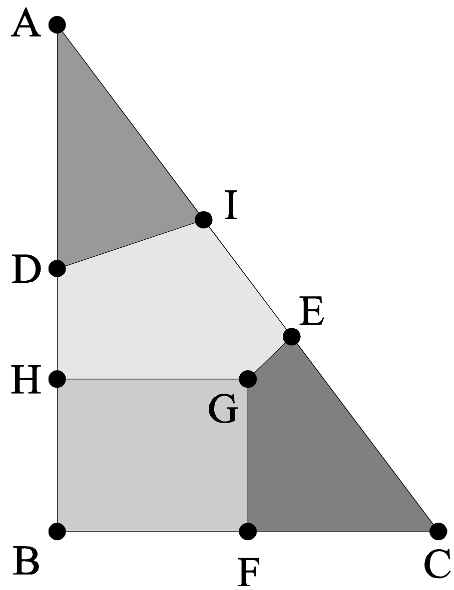
Setting this equal to t^2 yields

$$25 - 18t + \frac{18}{5}t^2 = t^2 \Rightarrow t = \frac{25}{13}, 5$$

Thus, A is distance $\frac{25}{13}$ from D:$= \left(0, \frac{27}{13}\right) \approx (0, 2.0769)$ which is at distance $\frac{25}{13}$ from E:$= \left(3-\frac{3}{5}\left(\frac{25}{13}\right), \frac{4}{5}\left(\frac{25}{13}\right)\right) = \left(\frac{24}{13}, \frac{20}{13}\right) \approx (1.8462, 1.5385)$. It is easy to verify that $\frac{25}{13}$ is the minimum distance between pairs of the five points A,B,C,D,E. Thus, $\frac{25}{13}$ is a lower bound on the diameter of any dissection of ABC into 4 regions. We try to show that this lower bound is realized. Let F$= \left(\frac{3}{2}, 0\right)$ and G$= \left(\frac{3}{2}, h\right)$ where $h > 0$ is chosen so that BG$=$CG$= \frac{25}{13}$.

$$\left(\frac{39}{26}\right)^2 + h^2 = \left(\frac{25}{13}\right)^2$$

We find $h = \frac{1}{26}\sqrt{979}$. Thus, G$= \left(\frac{3}{2}, \frac{1}{26}\sqrt{979}\right) \approx (1.5, 1.2034)$. Let H$= \left(0, \frac{1}{26}\sqrt{979}\right)$ and let I$= \left(3-\frac{3}{5}\left(5-\frac{25}{13}\right), \frac{4}{5}\left(5-\frac{25}{13}\right)\right) = \left(\frac{15}{13}, \frac{32}{13}\right) \approx (1.1538, 2.4615)$ be the point on AC at distance $\frac{25}{13}$ from A. Then BG$= \frac{25}{13}$, CG$= \frac{25}{13}$.



Now, the triangle ADI and the two quadrilaterals BFGH and FGEC clearly have diameter $\frac{25}{13}$. For the pentagon HGEID, we need to check that HE$\leq \frac{25}{13}$, but the height of E is less than that of the midpoint of DH since

$$1.538\,5 \approx \frac{20}{13} < \tfrac{1}{2}\left(\frac{1}{26}\sqrt{979} + \frac{27}{13}\right) \approx 1.6402.$$

Thus, all 4 regions have diameter equal to the lower bound $\frac{25}{13}$.