Putnam 1998 (Problems and Solutions)

A1. A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube.

Solution. Consider the cross section obtained by slicing vertically by a plane that contains a diagonal of the base of the cube. We obtain a rectangle of height $s =$side-length of the cube and width $\sqrt{2}s$. By similar triangles

$$\begin{aligned} \frac{3}{1} &= \frac{s}{1-\frac{1}{2}\sqrt{2}s} \Rightarrow \\ s &= \frac{6}{3\sqrt{2}+2} = \frac{6\left(3\sqrt{2}-2\right)}{\left(3\sqrt{2}+2\right)\left(3\sqrt{2}-2\right)} = \frac{9\sqrt{2}-6}{7}. \end{aligned}$$

A2. Let s be any arc of the unit circle lying entirely in the first quadrant. Let A be the area of the region lying below s and above the x-axis and let B be the area of the region lying to the right of the y-axis and to the left of s. Prove that $A+B$ depends only on the arc length of s and not on the position of s.

Solution. Let s run from θ_1 to θ_2. Then

$$\begin{aligned} A &= \int_{\cos\theta_2}^{\cos\theta_1} \sqrt{1-x^2}\,dx = -\int_{\theta_2}^{\theta_1} \sin^2\theta\,d\theta \\ B &= \int_{\sin\theta_1}^{\sin\theta_2} \sqrt{1-y^2}\,dy = \int_{\theta_1}^{\theta_2} \cos^2\theta\,d\theta \end{aligned}$$

Note that

$$\tfrac{\partial}{\partial\theta_1}(A+B) = -\sin^2\theta_1 - \cos^2\theta_1 = -1 \Rightarrow A+B = -\theta_1 + f(\theta_2)$$

while

$$\tfrac{\partial}{\partial\theta_2}(A+B) = \sin^2\theta_2 + \cos^2\theta_2 = 1 \Rightarrow A+B = \theta_2 + g(\theta_1).$$

Hence, since $A+B=0$ when $\theta_1=\theta_2$,

$$A+B = \theta_2 - \theta_1 + C = \theta_2 - \theta_1.$$

A3. Let f be a real function on the real line with continuous third derivative. Prove that there exists a point a such that

$$f(a)\cdot f'(a)\cdot f''(a)\cdot f'''(a) \geq 0.$$

Solution. Assume otherwise. Then

$$f(x)\cdot f'(x)\cdot f''(x)\cdot f'''(x)<0.$$

In particular, each of the factors is never zero. By replacing $f(x)$ by $-f(x)$ if necessary, we may assume that $f(x)>0$, and by replacing $f(x)$ by $f(-x)$ if necessary, we may assume that $f'(x)>0$. There are two cases:

Case 1: $f''(x)>0$ and $f'''(x)<0$.

Since $f'''(x)<0$, the graph of $f'(x)$ is concave down and hence the graph of $f'(x)$ lies below its tangent line at $x=0$. Thus,

$$f'(x)\le f'(0)+f''(0)\,x \quad\text{and}\quad f'\left(\frac{-f'(0)}{f''(0)}\right)\le 0 \text{ (contradiction).}$$

Case 2: $f''(x)<0$ and $f'''(x)>0$.

Since $f''(x)<0$, the graph of $f(x)$ is concave down and hence the graph of $f(x)$ lies below its tangent line at $x=0$. Thus,

$$f(x)\le f(0)+f'(0)\,x \quad\text{and}\quad f\left(\frac{-f(0)}{f'(0)}\right)\le 0 \text{ (contradiction).}$$

A4.

Let $A_1=0$ and let $A_2=1$. For $n>2$, the number A_n is defined by concatenating the decimal expansions of A_{n-1} and A_{n-2} from left to right. For example $A_3=A_2A_1=10$, $A_4=A_3A_2=101$, $A_5=A_4A_3=10110$, and so forth. Determine all n such that 11 divides A_n.

Solution. Since $10^n\equiv(-1)^n \bmod 11$, an integer is divisible by 11 iff the alternating sum of its digits (from right to left so that the first term is positive) is 0 mod 11. Let e_n be the alternating sum of the digits mod 11 of A_n from right to left, and let F_n be the number digits in A_n. Of course F_n is just the Fibonacci sequence. Note that $e_{n+2}=(-1)^{F_n}e_{n+1}+e_n$. We have $F_{n+2}=F_{n+1}+F_n$ and $F_1=1$, and $F_2=1$, we see that F_n is even iff n is divisible by 3. We have (all mod 11)

$$\begin{aligned}
e_{n+2} &= (-1)^{F_n}e_{n+1}+e_n\\
e_{3k+3} &= (-1)^{F_{3k+1}}e_{3k+2}+e_{3k+1}=-e_{3k+2}+e_{3k+1}\\
e_{3k+4} &= (-1)^{F_{3k+2}}e_{3k+3}+e_{3k+2}=-e_{3k+3}+e_{3k+2}=-\left(-e_{3k+2}+e_{3k+1}\right)+e_{3k+2}\\
&= -e_{3k+1}+2e_{3k+2}\\
e_{3k+5} &= (-1)^{F_{3k+3}}e_{3k+4}+e_{3k+3}=e_{3k+4}+e_{3k+3}=\left(-e_{3k+1}+2e_{3k+2}\right)+\left(-e_{3k+2}+e_{3k+1}\right)\\
&= e_{3k+2}\\
e_{3k+6} &= (-1)^{F_{3k+4}}e_{3k+5}+e_{3k+4}=-e_{3k+5}+e_{3k+4}=-e_{3k+2}+\left(-e_{3k+1}+2e_{3k+2}\right)\\
&= e_{3k+2}-e_{3k+1}\\
e_{3k+7} &= (-1)^{F_{3k+5}}e_{3k+6}+e_{3k+5}=-e_{3k+6}+e_{3k+5}=-\left(e_{3k+2}-e_{3k+1}\right)+e_{3k+2}=e_{3k+1}\\
e_{3k+8} &= (-1)^{F_{3k+6}}e_{3k+7}+e_{3k+6}=e_{3k+7}+e_{3k+6}=e_{3k+1}+e_{3k+2}-e_{3k+1}=e_{3k+2}\\
e_{3k+9} &= (-1)^{F_{3k+7}}e_{3k+8}+e_{3k+7}=-e_{3k+8}+e_{3k+7}=-e_{3k+2}+e_{3k+1}=e_{3k+3}
\end{aligned}$$

Thus, e_n is periodic of period 6. Now

$$\begin{aligned} e_1 &= 0,\ e_2 = 1,\ e_3 = -e_2 + e_1 = -1, \\ e_4 &= -e_1 + 2e_2 = 2, e_5 = e_2 = 1,\ e_6 = e_2 - e_1 = 1. \end{aligned}$$

Thus, e_n is divisible by 11 iff $n \equiv 1 \bmod 6$.

A5.

Let $\mathcal{F}$ be a finite collection of open disks in $\mathbb{R}^2$ whose union contains a set $E \subseteq \mathbb{R}^2$. Prove that there is a pairwise disjoint collection $D_1, ..., D_n$ in $\mathcal{F}$ such that

$$\bigcup_{j=1}^{n} 3D_j \supseteq E.$$

Here, if D is the disc of radius r and center P, then $3D$ is the disc of radius $3r$ and center P.

Solution. Note that if D and D' are open disks of radius r and r', with $r \leq r'$, then $D \cap D' \neq \phi \Rightarrow 3D' \supseteq D$. Starting with a largest disk, color it (say, red), and remove the disks which intersect it. The 3-fold enlargement of the colored disk together with the remaining disks will still cover E, by the above. Then color a largest remaining uncolored disk, and remove the remaining uncolored disks which intersect it. Continuing, the process eventually stops, since there are a finite number of disks. Moreover, at each stage the 3-fold enlargements of the colored disks and the remaining uncolored disks cover E. At the end of the process, no uncolored disks remain, and the 3-fold enlargements of the colored disks cover E. The colored disks are pairwise disjoint by construction.

A6.

Let A, B and C denote distinct points with integer coordinates in $\mathbb{R}^2$. Prove that if

$$(|AB| + |BC|)^2 < 8 \cdot [ABC] + 1$$

then A, B, C are three vertices of a square. Here $|XY|$ is the length of the segment XY and $[ABC]$ is the area of triangle ABC.

Solution. We have $[ABC] = \frac{1}{2}|AB|\ |BC| \sin\theta$, where θ is the angle of triangle ABC at vertex B. Thus, $4 \cdot [ABC] \leq 2|AB|\ |BC|$ with equality only if $\theta = 90°$. Also $2|AB|\ |BC| \leq |AB|^2 + |BC|^2$ with equality only if $|AB| = |BC|$, since $0 \leq (|AB| - |BC|)^2$. Hence,

$$\begin{aligned} 8 \cdot [ABC] &\leq 4 \cdot [ABC] + 2|AB|\ |BC| \\ &\leq 4 \cdot [ABC] + |AB|^2 + |BC|^2 \\ &\leq 2|AB|\ |BC| + |AB|^2 + |BC|^2 \\ &= (|AB| + |BC|)^2 < 8 \cdot [ABC] + 1 \end{aligned}$$

The intermediate expression $4 \cdot [ABC] + |AB|^2 + |BC|^2$ is an integer, since $2 \cdot [ABC]$ is a determinant of a 2×2 matrix with integer coefficients. Thus, we have all equalities, and $\theta = 90°$ and $|AB| = |BC|$.

B1. Find the minimum value of

$$\frac{(x+1/x)^6-(x^6+1/x^6)-2}{(x+1/x)^3+(x^3+1/x^3)}$$

Solution. We have

$$\begin{aligned}\frac{(x+1/x)^6-(x^6+1/x^6)-2}{(x+1/x)^3+(x^3+1/x^3)} &= \frac{(x+1/x)^6-(x^3+1/x^3)^2}{(x+1/x)^3+(x^3+1/x^3)} \\ &= \frac{\left((x+1/x)^3-(x^3+1/x^3)\right)\left((x+1/x)^3+(x^3+1/x^3)\right)}{(x+1/x)^3+(x^3+1/x^3)} \\ &= (x+1/x)^3-(x^3+1/x^3) \\ &= 3(x+1/x)\end{aligned}$$

$$\tfrac{d}{dx}(x+1/x)=\frac{x^2-1}{x^2}=0,\ x>0\Rightarrow x=1.$$

As $x+1/x$ approaches ∞ as $x\to+\infty$ and $x\to 0^+$, there is a minimum and the only candidate is $x=1$. The minimum value is $3(1+1/1)=6$.

B2. Given a point (a,b) with $0<b<a$, determine the minimum perimeter of a triangle with one vertex at (a,b) one on the x-axis and one on the line $y=x$. You may assume that a triangle with minimum perimeter exists.

Solution. Note that the distance of any point (d,d) on the line $y=x$ to the point (a,b) is the same as the distance of (d,d) to (b,a), the reflection of (a,b) in the line $y=x$. Also, the distance of any point $(c,0)$ to (a,b) is the same as the distance of $(c,0)$ to $(a,-b)$, the reflection of (a,b) in the x-axis. Thus, the perimeter of the triangle (d,d), (a,b), $(c,0)$ is the same as the broken line segment with vertices $(a,-b)$, $(c,0)$, (d,d), (b,a). The length of this broken line segment is no greater than the distance between the endpoints (b,a) and $(a,-b)$, namely $\sqrt{(b-a)^2+(a+b)^2}=\sqrt{2(a^2+b^2)}$. We can choose c and d so that the broken segment $(a,-b)$, $(c,0)$, (d,d), (b,a) is straight. Thus, $\sqrt{2(a^2+b^2)}$ is the minimum possible perimeter.

B3. Let H be the unit hemisphere $\{(x,y,z): x^2+y^2+z^2=1, z\geq 0\}$, C the unit circle $\{(x,y,0): x^2+y^2=1\}$, and P the regular pentagon inscribed in C. Determine the surface area of that portion of H lying over the planar region inside P, and write your answer in the form $A\sin\alpha+B\cos\beta$, where A,B,α,β are real numbers.

Solution. The desired area A is $\frac{1}{2}$ the area of the sphere minus 5 polar caps that each extends an angle of $\frac{\pi}{5}$ from its pole. Thus,

$$\begin{aligned} A &= \frac{1}{2}\left(4\pi - 5\int_0^{2\pi}\int_0^{\frac{\pi}{5}} \sin\varphi \, d\varphi d\theta\right) \\ &= 2\pi - 5\cdot\pi\left(-\cos\frac{\pi}{5}+1\right) \\ &= -3\pi + 5\pi\cos\frac{\pi}{5} = -3\pi\sin\frac{\pi}{2} + 5\pi\cos\frac{\pi}{5}. \end{aligned}$$

B4. Find necessary and sufficient conditions on positive integers m and n so that

$$\sum_{i=0}^{mn-1} (-1)^{\lfloor i/m\rfloor + \lfloor i/n\rfloor} = 0.$$

Solution. The number of terms in the sum is mn which is odd if both m and n are odd. Thus, the sum (consisting of 1s and -1s) cannot be zero if both m and n are odd. Suppose that $m+n$ is odd (e.g., m is even and n is odd). In this case we claim

$$(\lfloor i/m\rfloor + \lfloor i/n\rfloor) + (\lfloor (mn-1-i)/m\rfloor + \lfloor (mn-1-i)/n\rfloor) = m+n-2 \quad (1)$$

which is odd, so that for $0 \le i \le \frac{1}{2}mn$, the i-th term and the $(mn-1-i)$-th term cancel in the sum which is then 0. Note that

$$\begin{aligned} \lfloor i/m\rfloor + \lfloor (mn-1-i)/m\rfloor &= \lfloor i/m\rfloor + \lfloor n-(1+i)/m\rfloor \\ &= n + \lfloor i/m\rfloor + \lfloor -(1+i)/m\rfloor \\ &= n + (\lfloor i/m\rfloor + \lfloor -i/m - 1/m\rfloor) \\ &= n-1, \end{aligned} \quad (2)$$

and similarly

$$\lfloor i/n\rfloor + \lfloor (mn-1-i)/n\rfloor = m-1. \quad (3)$$

Adding (2) and (3), we get (1). Suppose that m and n are both even, say $m = 2m'$ and $n = 2n'$. Then, for any positive integer i,

$$\left\lfloor \frac{2i}{m}\right\rfloor = \left\lfloor \frac{2i}{2m'}\right\rfloor = \left\lfloor \frac{2i+1}{2m'}\right\rfloor \text{ and } \left\lfloor \frac{2i}{n}\right\rfloor = \left\lfloor \frac{2i}{2n'}\right\rfloor = \left\lfloor \frac{2i+1}{2n'}\right\rfloor.$$

Thus the sum, say $S(m,n)$, is given by twice the sum over even indices:

$$\begin{aligned}
S(m,n) &= \sum_{i=0}^{mn-1} (-1)^{\lfloor i/m \rfloor + \lfloor i/n \rfloor} = 2 \sum_{i'=0}^{\frac{1}{2}mn-1} (-1)^{\lfloor 2i'/m \rfloor + \lfloor 2i'/n \rfloor} \\
&= 2 \sum_{i'=0}^{\frac{1}{2}mn-1} (-1)^{\lfloor i'/m' \rfloor + \lfloor i'/n' \rfloor} = 2 \sum_{i'=0}^{2m'n'-1} (-1)^{\lfloor i'/m' \rfloor + \lfloor i'/n' \rfloor} \\
&= 2 \sum_{i'=0}^{m'n'-1} (-1)^{\lfloor i'/m' \rfloor + \lfloor i'/n' \rfloor} + 2 \sum_{i'=m'n'}^{2m'n'-1} (-1)^{\lfloor i'/m' \rfloor + \lfloor i'/n' \rfloor} \\
&= 2S(m',n') + 2 \sum_{i'=0}^{m'n'-1} (-1)^{\lfloor (i'+m'n')/m' \rfloor + \lfloor (i'+m'n')/n' \rfloor} \\
&= 2S(m',n') \left(1 + (-1)^{n'+m'}\right).
\end{aligned}$$

Now, if $1 + (-1)^{n'+m'} = 0 \Leftrightarrow n' + m'$ is odd, in which case $S(m',n') = 0$ and $S(m,n) = 0$. Thus, $S(m,n) = 0 \Leftrightarrow S(m',n') = 0 \Leftrightarrow \ldots \Leftrightarrow$ the highest power of 2 which divides m differs from the highest power of 2 which divides n.

B5.

Let N be the positive integer with 1998 decimal digits, all of them 1; that is,

$$N = 1111 \cdots 11.$$

Find the thousandth digit after the decimal point of $\sqrt{N}$.

Solution. Note that $9N = 1111 \cdots 11 = 10^{1998} - 1$. Thus,

$$\begin{aligned}
\sqrt{N} &= \sqrt{\frac{10^{1998}-1}{9}} = \tfrac{1}{3}\sqrt{10^{1998}-1} = \tfrac{1}{3}10^{999}\sqrt{1-10^{-1998}} \\
&= \tfrac{1}{3}10^{999}\left(1-10^{-1998}\right)^{\frac{1}{2}}
\end{aligned}$$

We have the binomial series

$$(1-x)^{\frac{1}{2}} = 1 - \tfrac{1}{2}x + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}x^2 - \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}x^3 + \cdots$$

valid for $|x| < 1$. The remainder term in $(1-x)^{\frac{1}{2}} = 1 - \frac{1}{2}x + R_2(x)$ satisfies

$$\begin{aligned}
|R_2(x)| &\le \max_{0 \le t \le x} \left| \tfrac{d^2}{dt^2}\left((1-t)^{\frac{1}{2}}\right) \right| \frac{x^2}{2!} = \max_{0 \le t \le x} \left| \tfrac{1}{2}\left(\tfrac{1}{2}-1\right)(1-t)^{-\frac{3}{2}} \right| \frac{x^2}{2!} \\
&= \left| \tfrac{1}{4}(1-x)^{-\frac{3}{2}} \right| \frac{x^2}{2!}
\end{aligned}$$

For $x = 10^{-1998}$, $(1-x)^{-\frac{3}{2}} \le 2$ and so $|R_2(x)| \le \frac{1}{4}x^2 \le \frac{1}{4}10^{-2\cdot 1998}$. Thus,

$$\left|\sqrt{N} - \tfrac{1}{3}10^{999}\left(1-\tfrac{1}{2}x\right)\right| = \tfrac{1}{3}10^{999}\left|(1-x)^{\frac{1}{2}} - \left(1-\tfrac{1}{2}x\right)\right|$$
$$\le \tfrac{1}{3}10^{999}\tfrac{1}{4}x^2 \le \tfrac{1}{12}10^{999-2\cdot 1998} = 10^{-2998}.$$

Now

$$\begin{aligned}
\tfrac{1}{3}10^{999}\left(1-\tfrac{1}{2}x\right) &= \tfrac{1}{3}10^{999}\left(1-\tfrac{1}{2}10^{-1998}\right) = \tfrac{1}{3}10^{999} - \tfrac{1}{6}10^{-999} \\
&= .3\overline{3} \times 10^{999} - .16\overline{6} \times 10^{-999} \\
&= 3\overset{999}{\cdots}3.3\overset{999}{\cdots}3 + \left(.33\overline{3} - .16\overline{6}\right) \times 10^{-999} \\
&= 3\overset{999}{\cdots}3.3\overset{999}{\cdots}3 + .16\overline{6} \times 10^{-999} \\
&= 3\overset{999}{\cdots}3.3\overset{999}{\cdots}316\overline{6}.
\end{aligned}$$

Thus, the 1000-th digit to the right of the decimal is 1.

B6.

Prove that, for any integers a, b, c, there exists a positive integer n such that $\sqrt{n^3 + an^2 + bn + c}$ is not an integer.

Solution. We try to write the assumed perfect square $n^3 + an^2 + bn + c$ in the form $\left(n^{3/2} + dn^{1/2} + f\right)^2$:

$$\begin{aligned}
n^3 + an^2 + bn + c &= \left(n^{3/2} + d\, n^{1/2} + f\right)^2 \\
&= n^3 + 2n^2 d + 2\left(\sqrt{n}\right)^3 f + nd^2 + 2d\sqrt{n}f + f^2.
\end{aligned}$$

Choosing $d = \frac{1}{2}a$, and $f = \pm 1$, we then have, for n sufficiently large,

$$\left(n^{3/2} + \tfrac{1}{2}a\, n^{1/2} - 1\right)^2 < n^3 + an^2 + bn + c < \left(n^{3/2} + \tfrac{1}{2}a\, n^{1/2} + 1\right)^2.$$

If n is a perfect square, say $n = m^2$, then the extreme left and right are perfect squares and there is only one perfect square between them, namely $\left(n^{3/2} + \frac{1}{2}a\, n^{1/2}\right)^2$. Hence, if $n = m^2$ and $n^3 + an^2 + bn + c$ is a perfect square, then

$$n^3 + an^2 + bn + c = \left(n^{3/2} + \tfrac{1}{2}a\, n^{1/2}\right)^2 = n^3 + a\, n^2 + \tfrac{1}{4}a^2 n.$$

or

$$m^6 + am^4 + bm^2 + c = m^6 + a\, m^4 + \tfrac{1}{4}a^2 m^2$$

or

$$bm^2 + c = \tfrac{1}{4}a^2 m^2$$

For this to hold for all sufficiently large integers m, we must have $c = 0$ and $b = \frac{1}{4}a^2$. Thus,

$$n^3 + an^2 + bn + c = \left(n^{3/2} + \tfrac{1}{2}a\, n^{1/2}\right)^2 = \left(\sqrt{n}\left(n + \frac{a}{2}\right)\right)^2,$$

which is not a perfect square, unless n is a perfect square.